An elementary proof of the affine plank conjecture for three planks on a triangle

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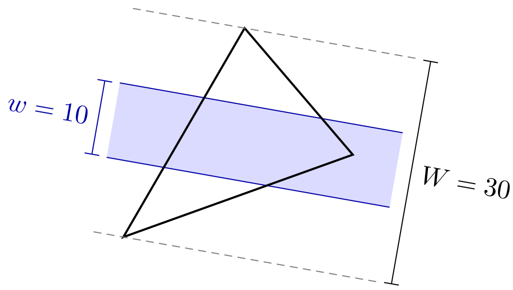
Abstract

We give an elementary proof that three planks covering a triangle have total relative width at least one. The proof uses shrinking, side rotation, and a comparison of the gaps on the sides.

1 The problem

A plank is the closed strip between two parallel lines. Its *relative width* is its width divided by the triangle's width in the same direction. The dashed lines are the triangle's supporting lines parallel to the plank. Here the relative width is $w/W = 1/3$.



Theorem. *If three planks P_1, P_2, P_3 cover a nondegenerate triangle T, their relative widths r_1, r_2, r_3 satisfy*

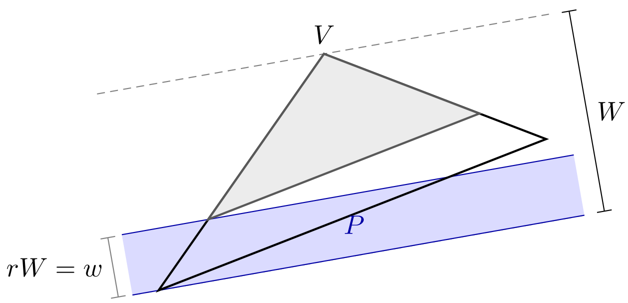
$$r_1 + r_2 + r_3 \geq 1.$$

Suppose, for a contradiction, that $S = r_1 + r_2 + r_3 < 1$.

2 No plank contains a vertex on a parallel supporting line

If a plank P of relative width $r < 1$ contains a vertex on a parallel supporting line, shrink T towards a vertex on the opposite supporting line by the factor $1 - r$. The other two planks cover the smaller triangle. This reduces to the two-plank case, whose bound gives total relative width at least $1 - r$ in T. Adding P gives $r + (1 - r) = 1$.

The two-plank case itself follows by the same argument: one plank must contain two vertices, at least one on a parallel supporting line. Removing it leaves a triangle covered by one plank, whose relative width is trivially at least 1.



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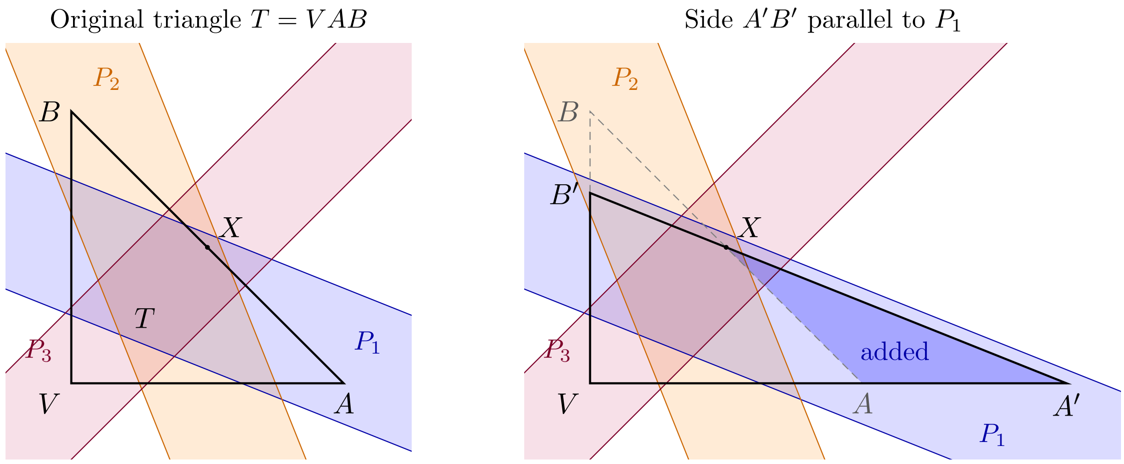
3 Two endpoint planks cannot cover a side

Each plank contains exactly one vertex: containing two would put one on a parallel supporting line, which Section 2 rules out. Label the vertices so that $A \in P_1$, $B \in P_2$, and $V \in P_3$. Suppose P_1 and P_2 together cover the whole side AB. Their intervals on AB must meet; choose a point X in their overlap.

Keep the planks fixed. Pivot the line AB about X until it is parallel to P_1, letting its intersections A', B' with the rays VA, VB slide. The new triangle $VA'B'$ is still covered. Its added piece, AXA', lies in P_1: $A, X \in P_1$, and the line through X parallel to the plank lies entirely in it.



Pivoting the other way, until the moving side is parallel to P_2, gives another covered triangle. Each new triangle has a whole side in a plank, so the shrinking argument gives total relative width at least one for both. We now show that the original total is at least the smaller of these two totals.

Take $V = (0,0)$, $A = (1,0)$, $B = (0,1)$ and write $X = (\xi, \eta)$. A rotated triangle has vertices

$$(0,0), \quad (1/u, 0), \quad (0, 1/v), \qquad \xi u + \eta v = 1.$$

Thus the reciprocal intercepts (u, v) vary along a line segment containing $(1,1)$. In the direction of P_1, the extreme vertices remain V and the moving point on VB. The triangle's width is divided by v, so P_1's relative width becomes $r_1 v$. Similarly, P_2's becomes $r_2 u$.

Write the direction of P_3 as $-cx + (1-c)y$, where $0 < c < 1$. Its range on the rotated triangle has length $c/u + (1-c)/v$. By the weighted harmonic–arithmetic mean inequality, the total relative width satisfies

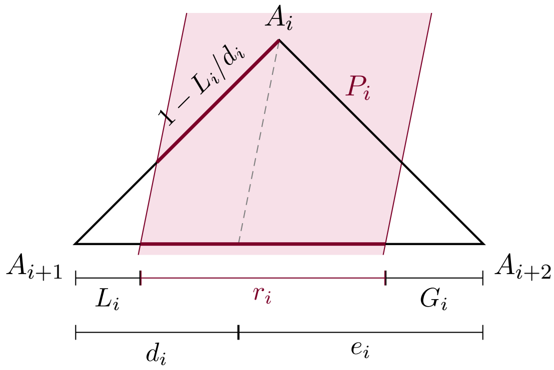
$$\begin{aligned} F(u,v) &= r_1 v + r_2 u + \frac{r_3}{c/u + (1-c)/v} \\ &\le r_1 v + r_2 u + r_3\big(cu + (1-c)v\big) =: L(u,v). \end{aligned}$$

At both endpoints, $F \ge 1$, hence $L \ge 1$. Since L is linear, it is at least one along the whole segment. But $L(1,1) = S < 1$, a contradiction.

4 Every side has a gap

We are left with three planks P_i containing vertices A_i, with indices read cyclically, such that each side has a gap between its endpoint planks. The opposite plank must bridge it.

On the side $A_{i+1}A_{i+2}$, let L_i, G_i be the fractions left outside P_i at A_{i+1}, A_{i+2}, respectively. The line through A_i parallel to P_i divides that side into fractions $d_i, e_i > 0$, where $d_i + e_i = 1$. Thus $r_i = 1 - L_i - G_i$.



By similar triangles, P_i covers a fraction $1 - L_i/d_i$ of A_iA_{i+1} from A_i. The bridging plank P_{i+2} starts at fraction L_{i+2}, no later than this endpoint. The same argument in the other direction gives

$$L_i + d_iL_{i+2} \le d_i, \qquad G_i + e_iG_{i+1} \le e_i.$$

To compare the two triples of gaps, put

$$k = 2 - \sum_i L_i, \qquad H_i = \frac{1 - L_i - L_{i+2}}{k}.$$

The lower inequalities give $L_i + L_{i+2} < 1$, so $k > 0$. Moreover,

$$H_i + e_iH_{i+1} - e_i = \frac{d_i - L_i - d_iL_{i+2}}{k} \ge 0.$$

Together with the upper inequalities, this gives $G_i + e_iG_{i+1} \le H_i + e_iH_{i+1}$.

Thus $\sum_i G_i \le \sum_i H_i$. Indeed, if some G_i exceeds H_i by $a > 0$, the next gap is below its comparison value by at least $a/e_i > a$, and the preceding gap is below its comparison value too. The total therefore decreases. If no gap exceeds its comparison value, the claim is immediate.

Since $\sum_i H_i = 2 - 1/k$, we obtain

$$S = 3 - \sum_i L_i - \sum_i G_i \ge k + \frac{1}{k} - 1 = 1 + \frac{(k-1)^2}{k} \ge 1,$$

the final contradiction.

Historical notes

The theorem is a case of Bang's affine plank conjecture [8]. Ball [7] proved the conjecture for centrally symmetric bodies; Verreault [14, Section 2.2.3] surveys the problem. Hunter [12] gave the earlier triangle claim and the final inequality for the configuration in his Figure 2. With no overlap on the sides, our final bound recovers his expression. The proof above supplies the geometric reductions and allows overlap.

The general two-plank bound is due to Bang [9]; see also Alexander [4, Theorem 4] and Gardner [11, Theorem 1]. The homothety used in Section 2 is a classical removal method, also described by Chambers and Mouillé [10, Section 1], who credit earlier work of Bang and Alexander.

Other sources informed the search for this proof: finite geometric reductions of Ohmann [13] and Ambrus [5, Section 3]; Gardner's relative width measures [11] and fractional coverings of Aharoni, Holzman, Krivelevich and Meshulam [1]; geometric replacement in Akopyan, Karasev and Petrov [3, Section 7]; Ball's matrix symmetrization [7, Lemma 4]; and Ambrus's contact-pair selection theorem [6, Theorem 2]. We also considered Alexander's homothet-avoidance equivalence [4, Theorem 2] and the planar convex-partition theorem of Akopyan and Karasev [2, Theorem 4.2].

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