The affine plank theorem for three planks in the plane

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Abstract

We prove that three planks covering a convex subset of the plane with nonempty interior have total relative width at least one. The planks may have parallel directions or zero width. The proof reduces a hypothetical counterexample to the convex hull of at most six points. Varying the plank boundaries then forces six supporting inequalities, whose slopes satisfy a planar compatibility condition. A cubic identity in the three widths completes the proof. We also give nonparallel equality examples and describe several consequences for equality.

1 Introduction

Bang's affine plank conjecture asks whether planks covering a convex body have total relative width at least one [9]. The relative width of a plank is its width divided by the width of the body in the same direction. We prove the conjectured inequality for three planks in the plane.

For a nonzero vector $v \in \mathbb{R}^2$, a *plank* is a set

$$P = \{x \in \mathbb{R}^2 : \alpha \le \langle v, x\rangle \le \beta\}, \qquad \alpha \le \beta.$$

If $C \subset \mathbb{R}^2$ is convex with nonempty interior, define

$$W_C(v) = \sup_{x,y\in C} \langle v, x - y\rangle, \qquad \mathrm{rw}(P, C) = \frac{\beta - \alpha}{W_C(v)}.$$

The factor $1/\|v\|$ in each Euclidean width cancels in this ratio. Nonempty interior gives $W_C(v) > 0$; when $W_C(v) = \infty$, we set $\mathrm{rw}(P, C) = 0$.

Theorem 1. *Let $C \subset \mathbb{R}^2$ be convex with nonempty interior. If three planks P_0, P_1, P_2 cover C, then*

$$\mathrm{rw}(P_0, C) + \mathrm{rw}(P_1, C) + \mathrm{rw}(P_2, C) \ge 1.$$

Parallel planks and planks of zero width are allowed. The constant is sharp: for any bounded C, divide the region between two parallel supporting lines into three consecutive planks. Appendix A gives nonparallel equality covers of a triangle with any prescribed positive relative widths summing to one.

Bang proved that the sum of the ordinary widths of covering planks is at least the minimum width of the body, and posed the affine strengthening [9]. Ball [8] established the affine inequality for centrally symmetric bodies. The case of coverings in at most two normal directions follows from Gardner's relative-width measures [12]; Akopyan, Karasev and Petrov [3, Section 7] give an elementary geometric proof. Hunter [13] treated the two-plank planar case and the three-plank triangle case, including an equality calculation. For arbitrarily many planks in $\mathbb{R}^d$, Bakaev and

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Yehudayoff [7] proved the lower bound $2/(1+\sqrt{d})$ for the sum of relative widths. Verreault [15, Section 2.2.3] surveys the subject.

Our proof starts from a cover whose total relative width is less than one. At most six points suffice to retain this strict inequality. Using the three plank directions as coordinates places their convex hull in a plane

$$x_0 + x_1 + x_2 = D, \qquad 0 \le x_i \le R_i.$$

The planks become coordinate intervals, which we call *windows.* The classical two-plank inequality implies that every window lies strictly inside its coordinate range.

There are eight possible sign patterns for a point outside all three windows. Planar convexity allows us to exclude each pattern by a supporting inequality involving two window endpoints, or by the plane equation itself. If fewer than six inequalities suffice, the six endpoints can move without increasing the total relative width until a window collapses or meets an endpoint of its range. This contradicts the two-plank inequality. A count on the cube of sign patterns, followed by the removal of redundant inequalities, leaves six supports: three exclude pairs of upper exterior halfplanes and three exclude pairs of lower exterior halfplanes.

The remaining step is numerical. Section 4 proves a width inequality for six supports satisfying a condition on their slopes. In Section 5, choosing each admissible slope as close to one as possible makes the plane equation imply that condition, completing the proof. Section 6 explains the relation to earlier methods. The appendices give equality examples, the finite catalogue of cap patterns, and further equality consequences.

2 Reduction to strict coordinate windows

We call a cover *deficient* if its total relative width is less than one. Throughout the proof of Theorem 1, we suppose that a deficient cover exists.

2.1 A finite convex hull and a positive plane equation

Choose finite numbers $0 < r_i < W_C(v_i)$ such that $\sum_i(\beta_i - \alpha_i)/r_i < 1$. For each i, choose $p_i, q_i \in C$ with $|\langle v_i, p_i - q_i\rangle| > r_i$. These choices are possible by the definition of the supremum, also when some directional widths are infinite. The convex hull of the six chosen points lies in C, is covered by the same planks, and remains deficient. Each of its three directional widths is positive. The hull need not have interior; only these positive directional ranges are used below.

This is the reduction to width-realizing points described by Ambrus [5, Section 3, pp. 11–12]. Approximate extrema suffice because the deficit is strict. From now on we retain these same six generators through every affine change of coordinates; all coordinate extrema are attained among them.

Use the three plank normals as coordinate functionals. This follows Ohmann's coordinate reduction [14], as described by Hunter [13, p. 819]. Their linear parts are dependent. Reflect coordinates as necessary so that the coefficients in a nonzero dependence are nonnegative. Normalize each coordinate range to $[0, 1]$. We then have

$$\sum_i A_i x_i = D, \qquad A_i \ge 0, \qquad \sum_i A_i > 0.$$

A coordinate interval $[\ell_i, h_i]$ represents the corresponding plank. Clipping both endpoints to $[0, 1]$ preserves coverage and cannot increase the interval's length.

We may arrange that every coefficient is positive. If the current total window length is $S < 1$, choose $0 < \varepsilon < (1 - S)/12$ and set

$$y_i = x_i - \varepsilon \sum_j x_j, \qquad B_i = (1 - 3\varepsilon)A_i + \varepsilon \sum_j A_j.$$

Then $B_i > 0$ and $\sum_i B_i y_i = (1-3\varepsilon)D$. Since $0 \le \sum_j x_j \le 3$, the windows $[\ell_i - 3\varepsilon, h_i]$ cover the new image. Each new coordinate range has length at least $1-3\varepsilon$, as seen by evaluating at old contacts with $x_i = 1$ and $x_i = 0$. Thus the new total relative width is at most

$$\frac{S+9\varepsilon}{1-3\varepsilon} < 1.$$

This perturbation includes parallel and repeated plank directions.

Absorb the positive coefficients into the coordinates and subtract their minima. Renaming the coordinates, we arrive at a finite hull $K = \operatorname{conv} F$, $|F| \le 6$, satisfying

$$K \subset \prod_{i=0}^{2} [0, R_i], \qquad x_0 + x_1 + x_2 = D, \qquad R_i > 0. \tag{1}$$

For each i, points of F attain both $x_i = 0$ and $x_i = R_i$; we call them *coordinate contacts*. They need not be distinct. After again clipping the windows to their ranges, the remaining claim is

$$K \subset \bigcup_{i=0}^{2} \{\ell_i \le x_i \le h_i\} \quad \Longrightarrow \quad \sum_{i=0}^{2} \frac{w_i}{R_i} \ge 1, \qquad w_i = h_i - \ell_i. \tag{2}$$

Our hypothetical cover still violates this inequality. All subsequent motions of the window endpoints keep K fixed.

2.2 The two-window inequality

We use Bang's classical two-plank bound [10], in a form that also applies to affine coordinates on a convex set. Other proofs appear in Alexander [4, Theorem 4], Gardner [12, Theorem 1] and Hunter [13, pp. 819–820]. We include a proof to make the argument self-contained.

Lemma 2 (Two windows)**.** *Suppose two affine coordinates on a convex set have attained, positive ranges. If one window in each coordinate covers the set, the sum of their lengths divided by the corresponding range lengths is at least one.*

Proof. Normalize the ranges to $[0,1]$, clip the windows to that interval, and translate so that they are $[0,w]$ and $[0,h]$. Suppose $w + h < 1$. Four coordinate contacts of the convex image are

$$L = (-a, s_0), \quad R = (w+b, t_0), \quad B = (p_0, -c), \quad T = (q_0, h+d),$$

where $a, b, c, d \ge 0$, $a + b = 1 - w$ and $c + d = 1 - h$. Let s, t be s_0, t_0 clipped to $[0,h]$, and let p, q be p_0, q_0 clipped to $[0,w]$. Coverage of the four segments joining horizontal to vertical contacts gives

$$ac \le ps, \qquad bc \le (w-p)t, \qquad ad \le q(h-s), \qquad bd \le (w-q)(h-t). \tag{3}$$

For example, if $ac > 0$, coverage of L and B puts $s_0 \in [0,h]$ and $p_0 \in [0,w]$, so no clipping occurs. Along LB, entering the first window must precede leaving the second. Hence $a/(a+p) \le s/(s+c)$, which gives $ac \le ps$. If $ac = 0$, that inequality is automatic. Reflections give the other three inequalities.

By Cauchy–Schwarz,

$$\sqrt{c}\,(a+b) \le \sqrt{p}\,\sqrt{as} + \sqrt{w-p}\,\sqrt{bt} \le \sqrt{w(as+bt)}.$$

Applying the same argument to the upper contacts yields

$$c(a+b)^2 \le w(as+bt), \qquad d(a+b)^2 \le w\big(a(h-s) + b(h-t)\big). \tag{4}$$

Add these inequalities and cancel $a + b > 0$. We obtain $(1-w)(1-h) \le wh$, or $w + h \ge 1$, a contradiction. $\square$

2.3 Excluding boundary windows

The next argument uses the homothetic removal of a boundary plank, as in Chambers and Mouillé [11, Section 1], who credit earlier work of Bang and Alexander.

Lemma 3 (Strict windows)**.** *In a deficient cover of K by windows inside its coordinate ranges,*

$$0 < \ell_i < h_i < R_i \qquad (i = 0, 1, 2).$$

Proof. A window of zero width can be removed. Indeed, any point of K on its level hyperplane is the limit of points on a segment to a coordinate contact off that hyperplane. Those points belong to the other two windows. Their union is closed, so it also contains the limit. Removing the zero-width window would contradict Lemma 2.

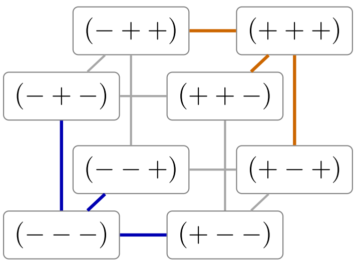
Suppose a window has the form $[0, h_i]$. Write $b = h_i/R_i$, and let c be the sum of the other two relative widths. Choose $b < t < 1 - c$ and a contact $P \in K$ with $P_i = R_i$. The homothetic copy $tP + (1-t)K$ lies in K and strictly beyond the first window. The other two windows cover this copy, on which their total relative width is $c/(1-t) < 1$, again contradicting Lemma 2. Reflection treats a window ending at R_i. ☐

3 Six necessary supporting inequalities

3.1 The cube of exterior signs

A point outside all three windows has, in each coordinate, sign $+$ if $x_i > h_i$ and sign $-$ if $x_i < \ell_i$. These eight sign patterns form the vertices of a cube. We call the intersection of K with one exterior halfplane a *cap*, and the intersection with two exterior halfplanes a *pair cap*. An empty pair cap excludes the two sign patterns on the corresponding cube edge.

For example, the edge from $+--$ to $+-+$ represents the pair cap $\{x_0 > h_0,\ x_1 < \ell_1\} \cap K$. The three lower–lower pairs are the edges at $---$, and the three upper–upper pairs are the edges at $+++$, highlighted below.



The plane equation can itself exclude either constant sign pattern:

$$\sum_i \ell_i \le D \quad (---), \qquad D \le \sum_i h_i \quad (+++). \tag{5}$$

We use either of these inequalities only when it holds.

Lemma 4 (Excluding a sign pattern)**.** *Every cube vertex is excluded by an empty incident pair cap or by the corresponding inequality in* (5)*.*

Proof. Let H_0, H_1, H_2 be the three open exterior halfplanes on $x_0 + x_1 + x_2 = D$ for the sign pattern under consideration. If their intersection is empty, the pattern is constant and its inequality in (5) holds. To see this, a mixed pattern always meets the plane: send the two coordinates with equal signs far in that direction and solve for the third. The all-lower and all-upper patterns meet the plane exactly when their respective inequalities fail.

Otherwise, choose $Q \in H_0 \cap H_1 \cap H_2$. If every pair cap meets K, choose $P_i \in K$ in the two halfplanes other than H_i. The four planar points Q, P_0, P_1, P_2 have an affine dependence. Separating its positive and negative coefficients and normalizing gives a point Z in the convex hulls of two disjoint nonempty groups of these points. Each of K, H_0, H_1, H_2 contains all four points except possibly one. For each set, one of the two groups omits that exceptional point, so convexity puts Z in the set. Thus $Z \in K \cap H_0 \cap H_1 \cap H_2$, contrary to coverage. □

3.2 From empty caps to affine constraints

Strictness ensures that each individual exterior cap contains a coordinate contact. Consequently an empty pair cap can be separated by an inequality with positive coefficients. The following separation argument identifies the admissible slopes.

Lemma 5 (Admissible slopes)**.** *Let g, h be affine functions on a convex set K. Suppose each of $\{g > 0\} \cap K$ and $\{h > 0\} \cap K$ is nonempty, while $\{g > 0, h > 0\} \cap K$ is empty. The set of $c > 0$ for which $g + ch \le 0$ on K is a nonempty closed bounded interval contained in $(0, \infty)$.*

Proof. If $p, q \in K$ satisfy $g(p) > 0$ and $h(q) > 0$, the segment pq avoids $\{g > 0, h > 0\}$. It follows that

$$h(p) < 0, \qquad g(q) < 0, \qquad g(p)h(q) \le g(q)h(p).$$

Indeed, failure of any of these conclusions would place a point of the segment in $\{g > 0, h > 0\}$. Hence

$$\frac{g(p)}{-h(p)} \le \frac{-g(q)}{h(q)}.$$

Take the supremum of the lower bounds on the left and the infimum of the upper bounds on the right. Both sets are nonempty, and every lower bound is at most every upper bound. They therefore define an interval $[c_-, c_+]$ with $0 < c_- \le c_+ < \infty$. Points with $g, h \le 0$ impose no further restriction, so this is exactly the admissible interval. □

When $K = \operatorname{conv} F$ with F finite, it suffices to impose the inequality at the points of F. The admissible interval is then determined by finitely many bounds. For a mixed pair, the conclusion takes the form

$$x_i - cx_j \le h_i - c\ell_j \quad (x \in K), \qquad c > 0. \tag{6}$$

Reflecting one coordinate gives the corresponding upper–upper or lower–lower inequality. Fixing c turns (6) into the single affine constraint

$$h_i - c\ell_j \ge \max_{x \in F}(x_i - cx_j)$$

on the six window endpoints. Thus any selected empty pair cap can be kept empty during endpoint motions by imposing one affine inequality. The inequalities in (5) are already affine in the endpoints.

3.3 Endpoint motion and the cube count

Write the endpoints as $B = (\ell_0, \ell_1, \ell_2, h_0, h_1, h_2)$. Total relative width is a linear function of B.

Lemma 6 (Six constraints are necessary)**.** *Suppose m affine inequalities hold at a strict deficient cover and guarantee coverage whenever the windows are strict. Then $m \ge 6$.*

Proof. If $m < 6$, the linear parts of the inequalities have a nonzero common kernel vector d. Choose its sign so that total relative width does not increase along $B + td$, $t \ge 0$. All m constraints remain satisfied. Move until the first boundary of the region $0 < \ell_i < h_i < R_i$; this takes finite time because that region is bounded.

At the limiting endpoints the windows remain ordered and lie inside their ranges. They still cover K: for each fixed point of K, membership in at least one of the three closed windows is preserved in the limit. Their total relative width is still less than one. This contradicts Lemma 3. $\square$

By Lemma 4, the available empty edges and the available constant vertices cover the cube. Choose an inclusion-minimal such cover. Its *selections* are the chosen edges and singleton vertices. A *full star* is the set of three edges incident to one vertex.

Lemma 7 (Cube count)**.** *An inclusion-minimal cover of the cube by edges and, possibly, its two constant vertices has at most six selections. If it has six, then either both constant vertices are selected, both constant full stars are selected, or a mixed full star is selected.*

Proof. Each selected edge has an endpoint belonging to no other selection. The selected edges therefore form disjoint stars, and selected singleton vertices are isolated. If there are c stars and $k \le 2$ singletons, the stars contain $8-k$ vertices and $8-k-c$ edges. There are $8-c$ selections. A star has at most four vertices, so $c \ge 2$.

Six selections mean $c = 2$. Unless $k = 2$, the stars contain at least seven vertices, so one is full. If its center is constant, its leaves cannot meet any other selected edge. The three vertices at distance two must then be covered by the opposite full star. Otherwise the full star has a mixed center. $\square$

Proposition 8 (The six same-sign pairs)**.** *In a deficient cover, all three lower–lower pair caps and all three upper–upper pair caps are empty.*

Proof. For each selection in a minimal cube cover, keep its pair cap empty by fixing an admissible support, or retain its plane inequality. These affine constraints preserve coverage. Lemmas 6 and 7 force exactly six selections.

Suppose first that both constant vertices are selected. Minimality keeps the four selected edges away from those vertices. They cover the six mixed vertices, so two edges meet. Both are mixed edges. A mixed full star also contains two incident mixed edges. After permuting coordinates and possibly reflecting all three, either situation gives the empty pair caps

$$\{x_0 > h_0,\ x_1 < \ell_1\} \cap K, \qquad \{x_0 > h_0,\ x_2 < \ell_2\} \cap K.$$

At any strict windows preserving these exclusions, a contact with $x_0 = R_0 > h_0$ has $x_1 \ge \ell_1$ and $x_2 \ge \ell_2$. The plane equation therefore gives

$$\ell_0 + \ell_1 + \ell_2 < D. \tag{7}$$

If both constant vertices were selected, their lower plane constraint is redundant by (7); delete it. If a mixed full star was selected, delete its lower–lower edge. One endpoint, $---$, is excluded by (7), and the other, $+--$, is excluded by the two mixed edges. In either case five affine constraints still preserve coverage throughout the strict-window region, contradicting Lemma 6.

The cube count leaves only the two constant full stars. Their six selected edges are exactly the six same-sign pairs. $\square$

4 A width inequality for six supports

We now isolate the numerical step. It uses only supporting inequalities and coordinate contacts; no convexity or plane equation is needed. In this and the next section, indices are cyclic, with $j = i+1$ and $k = i-1$ modulo three.

Lemma 9 (Six-support inequality)**.** *Let $R_i > 0$, and let $K \subset \mathbb{R}^3$ contain a point with $x_i = 0$ and a point with $x_i = R_i$ for each i. Suppose $\ell_i \le h_i$ and positive numbers a_i, b_i satisfy*

$$x_i + a_i x_j \le h_i + a_i h_j, \qquad x_i + b_i x_j \ge \ell_i + b_i \ell_j \quad (x \in K). \tag{8}$$

If, for every cyclic index,

$$a_j(1 - b_i) \le 1, \qquad b_j(1 - a_i) \le 1, \tag{9}$$

then $\sum_i (h_i - \ell_i)/R_i \ge 1$.

Proof. We first bound each coordinate range using the supports, and then combine the three bounds by a cubic identity.

Bounds from the coordinate contacts. Put $w_i = h_i - \ell_i$, $u_i = R_i - h_i$, and

$$D_{L,i} = 1 + b_i a_j b_k, \qquad D_{U,i} = 1 + a_i b_j a_k, \qquad E_i = D_{L,i} D_{U,i} > 0.$$

At a contact with $x_i = 0$, the lower inequality for (i, j), the upper one for (j, k) and the lower one for (k, i) give

$$\begin{aligned} \ell_i &\le b_i(x_j - \ell_j) \le b_i\big(w_j + a_j(h_k - x_k)\big) \\ &\le b_i\big(w_j + a_j w_k - a_j b_k \ell_i\big). \end{aligned}$$

Rearranging, and reflecting for the contact at $x_i = R_i$, we obtain

$$D_{L,i} \ell_i \le b_i(w_j + a_j w_k), \qquad D_{U,i} u_i \le a_i(w_j + b_j w_k). \tag{10}$$

Multiplying by $D_{U,i}$ and $D_{L,i}$ respectively, adding, and using $R_i = w_i + \ell_i + u_i$, we obtain

$$E_i R_i \le S_i = E_i w_i + F_i w_j + H_i w_k,$$

where

$$F_i = b_i D_{U,i} + a_i D_{L,i}, \qquad H_i = b_i a_j D_{U,i} + a_i b_j D_{L,i}.$$

All three coefficients in each S_i are positive.

The coefficient identities. The slope conditions give nonnegative numbers

$$q_i = 1 - a_j + a_j b_i, \qquad r_i = 1 - b_j + b_j a_i.$$

Define

$$G = \prod_{i=0}^{2} q_i + \prod_{i=0}^{2} r_i + \sum_{i=0}^{2} (a_k + b_k) q_i r_i \ge 0. \tag{11}$$

Writing $A = \prod_i a_i$, $B = \prod_i b_i$ and $\delta = AB - 1$, expansion gives

$$G = 2(A - 1)(B - 1) + \prod_i (a_i + b_i). \tag{12}$$

The definitions of E_i, F_i, H_i also give

$$\begin{gathered} E_i E_j - F_i H_j = \delta^2, \\ 2 \prod_i E_i - \prod_i F_i - \prod_i H_i = \delta^2 G. \end{gathered} \tag{13}$$

The cubic identity. Expand in w_0, w_1, w_2. The pure cubes cancel, and the remaining terms group as

$$\sum_i E_i w_i S_j S_k - \prod_i S_i = \sum_i (E_j E_k - F_j H_k)(S_i - E_i w_i) w_j w_k + \left(2\prod_i E_i - \prod_i F_i - \prod_i H_i\right) \prod_i w_i.$$

Substitution of (13) yields

$$\sum_i E_i w_i S_j S_k - \prod_i S_i = \delta^2 \left[\sum_i (S_i - E_i w_i) w_j w_k + G \prod_i w_i\right] \geq 0. \tag{14}$$

Every term on the right is nonnegative. Since $S_i \geq E_i R_i > 0$, division by $\prod_i S_i$ gives

$$1 \leq \sum_i \frac{E_i w_i}{S_i} \leq \sum_i \frac{w_i}{R_i}.$$

We divide by neither a window width nor δ, so the argument includes zero widths and $AB = 1$. $\square$

5 The planar slope condition

Return to the deficient cover of the finite hull in (1). Proposition 8 makes all six same-sign pair caps empty. By Lemma 5, each has an interval of admissible positive slopes. For each upper and lower support, choose the slope in this interval nearest to one, obtaining (8). We show that these choices satisfy (9). Figure 1 illustrates the two contacts used in the comparison.

Consider $a_j(1 - b_i) \leq 1$. It holds immediately if $a_j \leq 1$ or $b_i \geq 1$. Otherwise $a_j > 1$ and $b_i < 1$. Because a_j cannot decrease toward one, an inequality at some $P \in F$ must be an equality and prevent that decrease. Thus

$$P_j + a_j P_k = h_j + a_j h_k, \qquad P_j > h_j.$$

Similarly, increasing b_i toward one is prevented by a point $Q \in F$ with

$$Q_i + b_i Q_j = \ell_i + b_i \ell_j, \qquad Q_j < \ell_j.$$

These contacts exist because the admissible intervals are defined by finitely many inequalities.

Set

$$d = P_j - Q_j > 0, \qquad u = P_i - Q_i, \qquad v = Q_k - P_k.$$

Evaluate the upper support at Q and the lower support at P, then subtract the corresponding equalities. This gives

$$a_j v \leq d, \qquad -b_i d \leq u.$$

The plane equation gives $u + d = v$, and hence

$$a_j(1 - b_i) d \leq a_j(u + d) = a_j v \leq d.$$

Canceling $d > 0$ proves the required inequality. The reflection $x_i \mapsto R_i - x_i$ exchanges upper and lower supports and preserves the form of the plane equation. Applying the same argument proves $b_j(1 - a_i) \leq 1$.

Completion of the proof of Theorem 1. The chosen slopes satisfy the hypotheses of Lemma 9. It gives $\sum_i w_i/R_i \geq 1$, contradicting the deficient cover. The reductions of Section 2 turn any counterexample to the theorem into such a cover, so the theorem follows. $\square$

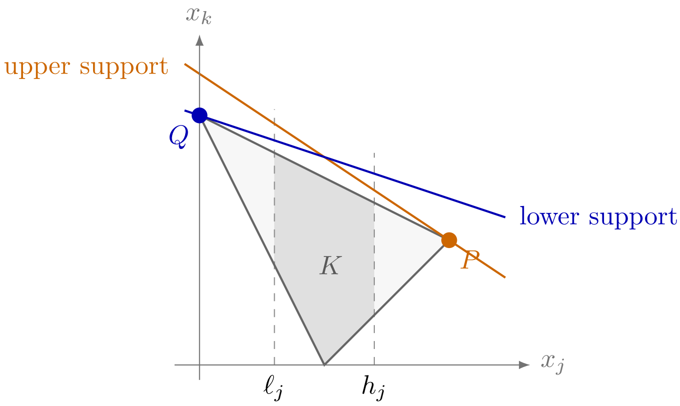


Figure 1: The contacts that prevent the two slopes from moving toward one. The upper support meets P with $P_j > h_j$, and the lower support meets Q with $Q_j < \ell_j$. In this example, $x_i + x_j + x_k = 3$, the vertices in (x_j, x_k) coordinates are $(2,1), (0,2), (1,0)$, all windows are $[3/5, 7/5]$, and the chosen slopes are $a_j = 3/2$ and $b_i = 2/3$.

6 Remarks on the method

The proof separates a planar geometric step from an algebraic inequality. Planarity enters when three exterior halfplanes are reduced to pair exclusions, and again when the relation $u + d = v$ compares the two supporting slopes. Lemma 9 itself needs only the six supports and their coordinate contacts. Thus the numerical inequality applies more generally, although the argument that produces its slope conditions uses the plane equation.

The finite reduction of Ambrus [5] and the replacement and deletion arguments of Akopyan, Karasev and Petrov [3, Section 7] guided this approach. Here fixing a support makes the dependence on the window endpoints affine. The endpoint motion and cube count reduce the covering problem to the six same-sign pairs.

Other formulations help explain the need to retain the positions of the windows. A probability measure with uniform projections in the three plank directions would prove the inequality at once, but Gardner's triangle example [12, Example 1] shows that such a measure need not exist. Aharoni, Holzman, Krivelevich and Meshulam [1] show that fractional plank coverings of the coordinate simplex can have cost below one. These obstructions guided the use of actual covering intervals. Ball's matrix symmetrization [8, Lemma 4] and Ambrus's contact-pair selection theorem [6, Theorem 2] also informed the search for supporting data. The latter requires a symmetry condition on the chosen pairs; our slope comparison instead follows from their common plane.

Two geometric reductions were useful to consider, but do not directly give the required fixed-count covering statement. Alexander [4, Theorem 2] relates the affine plank conjecture to homothetic copies of a convex body avoiding prescribed lines. His equivalence concerns all numbers of planks, and its converse uses subdivision; a statement about three lines alone does not imply the three-plank inequality with unequal widths. Akopyan and Karasev [2, Theorem 4.2] prove the analogous relative-inradius inequality for convex partitions in the plane. As they note in their Section 5, a plank cover need not contain such a partition. The proof here works directly with the covering condition.

A Equality covers of a triangle

The sharp constant in Theorem 1 can be attained with three distinct directions. Hunter's triangle calculation [13, pp. 820–821] gives an earlier study of equality. The construction below is parametrized by an interior point and realizes every positive triple of relative widths summing to one. Indices in this appendix are cyclic modulo three. Figure 2 shows one example.

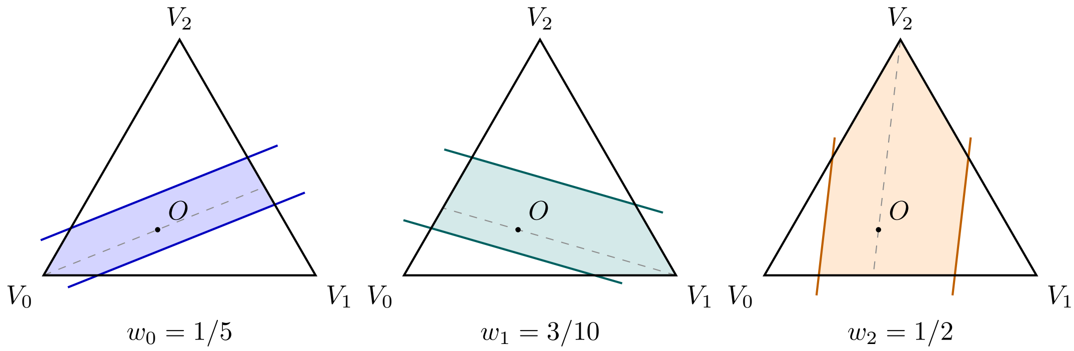


Figure 2: An equality cover with relative widths 1/5, 3/10 and 1/2. Each panel shows one plank. The chosen point O has barycentric coordinates $(15, 10, 6)/31$; the dashed line OV_i is parallel to the corresponding plank.

Proposition 10. *Let t_i be barycentric coordinates on a nondegenerate triangle with vertices V_i. Choose $p_i > 0$ with $\sum_i p_i = 1$, and set*

$$O = \sum_i p_i V_i, \qquad T = p_0p_1 + p_1p_2 + p_2p_0, \qquad y_i = \frac{Tt_i}{p_i}.$$

For $j = i + 1$ and $k = i - 1$, the three planks

$$P_i = \{-p_j \le y_j - y_k \le p_k\}$$

cover the triangle and have relative widths $w_i = p_jp_k/T$, whose sum is one. The boundaries of P_i are parallel to OV_i. The three directions are distinct, and each window lies strictly inside its directional range.

Proof. The range of $y_j - y_k$ is $[-T/p_k, T/p_j]$, so the relative width of P_i is

$$\frac{p_j + p_k}{T/p_j + T/p_k} = \frac{p_jp_k}{T}.$$

Since $T > p_jp_k$, the window endpoints are strictly inside the range. The affine function $y_j - y_k$ vanishes at both O and V_i. Its level lines are therefore parallel to OV_i, and the three directions are distinct because O is interior.

Suppose a point of the triangle misses all three planks. Relabel so that $y_0 \le y_1 \le y_2$; exchanging the two indices of a pair leaves its plank unchanged. Missing P_2 and P_0 now gives $y_1 > y_0 + p_0$ and $y_2 > y_1 + p_1$. Since $y_0 \ge 0$,

$$T = \sum_i p_iy_i > p_1p_0 + p_2(p_0 + p_1) = T,$$

a contradiction. □

For any prescribed $w_i > 0$ with $\sum_i w_i = 1$, choose

$$p_i = \frac{1/w_i}{1/w_0 + 1/w_1 + 1/w_2} = \frac{w_{i+1}w_{i-1}}{w_0w_1 + w_1w_2 + w_2w_0}. \tag{15}$$

Then $p_jp_k/T = w_i$. The map taking a positive normalized triple to its normalized reciprocals is an involution. Thus the interior point O and the relative widths give equivalent parametrizations of this family.

The equal-width case also illustrates the contacts considered in Appendix C. For $w_i = 1/3$, introduce range coordinates $x_i = (1 + t_{i+1} - t_{i-1})/2$. Then

$$K = \operatorname{conv}\{(0, 1, \tfrac{1}{2}), (\tfrac{1}{2}, 0, 1), (1, \tfrac{1}{2}, 0)\}, \qquad [\ell_i, h_i] = [\tfrac{1}{3}, \tfrac{2}{3}].$$

In the coordinates $z_i = 3x_i - 1$, the windows are $[0, 1]$ and $\sum_i z_i = 3/2$. If e_i are the coordinate unit vectors, the lower and upper pair contacts are

$$L_i = \tfrac{3}{2} e_i, \qquad U_i = (1, 1, 1) - \tfrac{3}{2} e_i.$$

Each lies on a side of the triangle, as shown in Figure 3.

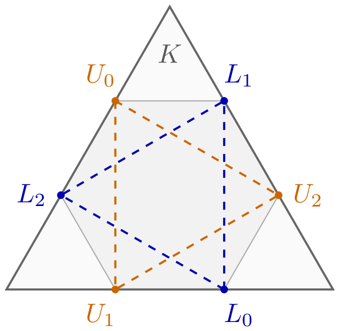


Figure 3: The six pair contacts when all relative widths are $1/3$. Each L_i fixes the other two coordinates at the lower window endpoints; each U_i fixes them at the upper endpoints. Each side of K contains two contacts, at one-third and two-thirds of its length. The dashed triangles join the lower contacts and the upper contacts, respectively.

B The six-cycle of cap patterns

Removing the two constant vertices from the sign cube leaves a cycle of length six. On this cycle, a same-sign pair cap excludes one vertex, while a mixed pair cap excludes two adjacent vertices. We call a selection of such vertices and edges a *cap pattern*. This is terminology for the finite catalogue below.

Number the mixed sign patterns in cyclic order as follows. The symbol S_n denotes the same-sign pair cap associated with vertex n, and E_n denotes the mixed pair cap associated with the edge from n to $n+1$, with indices modulo six. All the displayed sets are intersected with K.

Node	Signs	Same-sign pair cap S_n	Mixed pair cap E_n
0	$+--$	$x_1 < \ell_1,\ x_2 < \ell_2$	$x_0 > h_0,\ x_1 < \ell_1$
1	$+-+$	$x_0 > h_0,\ x_2 > h_2$	$x_2 > h_2,\ x_1 < \ell_1$
2	$--+$	$x_0 < \ell_0,\ x_1 < \ell_1$	$x_2 > h_2,\ x_0 < \ell_0$
3	$-++$	$x_1 > h_1,\ x_2 > h_2$	$x_1 > h_1,\ x_0 < \ell_0$
4	$-+-$	$x_0 < \ell_0,\ x_2 < \ell_2$	$x_1 > h_1,\ x_2 < \ell_2$
5	$++-$	$x_0 > h_0,\ x_1 > h_1$	$x_0 > h_0,\ x_2 < \ell_2$

A pattern is a cover of this cycle by selected vertices and edges. It is *minimal* if no selected element can be removed while preserving coverage. This combinatorial condition does not assert geometric realizability. In a covering argument, the two constant sign patterns must also be excluded.

Proposition 11. *Up to rotations and reflections of the six-cycle, the minimal cap patterns are the eight families in the following table. There are* 39 *labelled minimal patterns.*

Pattern	Selected vertices	Selected edges	Size	Orbit size
A	$0, 1, 2, 3, 4, 5$	none	6	1
B	$2, 3, 4, 5$	0	5	6
C	$3, 4, 5$	$0, 1$	5	6
D	$4, 5$	$0, 2$	4	6
E	$2, 5$	$0, 3$	4	3
F	5	$0, 1, 3$	4	12
G	none	$0, 2, 4$	3	2
H	none	$0, 1, 3, 4$	4	3
			Total	39

Proof. A selected vertex cannot be incident to a selected edge: it would be redundant. Every vertex incident to no selected edge must, however, be selected. Thus the edge set determines the selected vertices uniquely.

A selected edge is redundant exactly when both neighboring edges are selected. Admissible edge sets are therefore precisely those containing no three cyclically consecutive edges. Conversely, for any such edge set, adding exactly the uncovered vertices gives a minimal pattern: each selected element covers a vertex that no other selected element covers.

Classify these edge sets by size. Zero or one edge gives A or B, with one or six labelled choices. Two edges are adjacent, at distance two, or opposite, giving C, D or E, with six, six or three choices.

Three edges either alternate, giving the two patterns of G, or consist of an adjacent pair and an isolated edge. There are six choices of the pair and two positions for the isolated edge, giving the twelve patterns of F; reflection exchanges those two positions. With four selected edges, the two omitted edges must be opposite, giving the three patterns of H. Five or six selected edges necessarily include three consecutive edges. The list is exhaustive, and its sizes sum to

$$1 + 6 + 6 + 6 + 3 + 12 + 2 + 3 = 39.$$

$\square$

Figure 4 shows the representatives. Pattern A is the six same-sign pairs forced by Proposition 8; pattern C will be used in Appendix C.

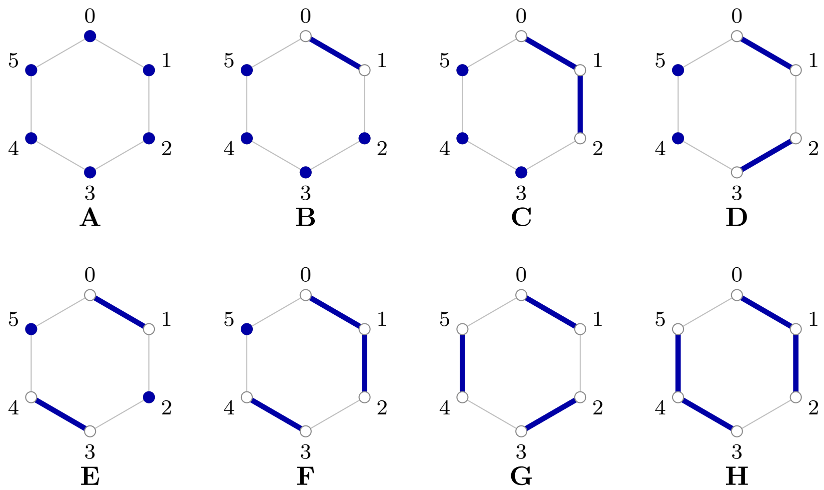


Figure 4: The eight minimal cap patterns on the cycle of mixed signs. A filled vertex selects the same-sign pair cap S_i, and a thick edge selects the mixed pair cap E_i. Every vertex is selected or incident to a selected edge. These are combinatorial diagrams, not drawings of the covered set.

C Consequences for equality

We record three consequences of the proof for equality covers. They describe equality in the numerical inequality, exclude one mixed cap pattern, and give an endpoint motion for covers with six empty same-sign pairs. They do not constitute a classification of all geometric equality covers.

For this purpose it is convenient to normalize each coordinate range to $[0, 1]$. Dividing the coordinates in (1) by R_i gives

$$K \subset [0,1]^3, \qquad \sum_i \lambda_i x_i = D, \qquad \lambda_i > 0, \tag{16}$$

where every coordinate attains both zero and one. Window lengths are now relative widths. Indices below are cyclic modulo three.

C.1 Equality in the six-support inequality

Use the notation of Section 4, and define the matrix N by

$$(Nx)_i = E_i x_i + F_i x_j + H_i x_k.$$

Its three principal 2×2 minors are the expressions $E_i E_j - F_i H_j$ in (13), so all equal δ^2. The products A, B and the expression G in (12) are unchanged by rescaling the coordinate ranges to one.

Proposition 12. *Suppose the six positive-slope supports (8) hold on a set meeting $x_i = 0$ and $x_i = 1$ for every i. If $G \geq 0$, every $w_i > 0$, and $\sum_i w_i = 1$, then*

$$AB = 1, \qquad S_i = E_i, \qquad N_{ij} = E_i \quad \textit{for all } i, j. \tag{17}$$

All six coordinate-contact inequalities in (10) are equalities.

Proof. Because $F_i, H_i, w_j, w_k > 0$, $S_i - E_i w_i = F_i w_j + H_i w_k > 0$. The bracket in (14) is therefore strictly positive. Equality in

$$1 \leq \sum_i \frac{E_i w_i}{S_i} \leq \sum_i w_i = 1$$

forces $\delta = 0$ and $S_i = E_i$ for every i.

Set $m_{ij} = N_{ij}/E_i$. The vanishing principal minors give $m_{ij} m_{ji} = 1$, while the equalities $S_i = E_i$ give $\sum_j m_{ij} w_j = 1$. Hence

$$\sum_{i,j} w_i w_j (m_{ij} + m_{ji} - 2) = 0. \tag{18}$$

Each summand is nonnegative, since a positive number and its reciprocal have sum at least two. All weights are positive, so $m_{ij} = 1$ for every pair. This proves (17). Finally, $S_i - E_i$ is a positive weighted sum of the two nonnegative differences in (10). Both differences must vanish. $\square$

The reciprocal-pair argument in (18) works in any finite matrix size: positive weights of total one, reciprocal positive entries, and row sums weighted by those weights equal to one force every entry to be one.

C.2 A mixed pattern forces strict inequality

Pattern C in Appendix B consists of the three same-sign caps S_3, S_4, S_5 and the two mixed caps E_0, E_1. The next proposition excludes equality for this pattern under a nondegenerate plane relation.

Proposition 13. *Let $K \subset [0,1]^3$ be convex and attain zero and one in every coordinate. Suppose $0 < \ell_i < h_i < 1$ and the five caps S_3, S_4, S_5, E_0, E_1 are empty. If K lies in a plane $A_0x_0 + A_1x_1 + A_2x_2 = D$ with $A_1 \neq 0$, then $\sum_i (h_i - \ell_i) > 1$.*

Proof. Lemma 5 gives positive numbers a_0, a_1, b, c, d such that

$$\begin{aligned} x_0 + a_0x_1 &\le h_0 + a_0h_1, & x_1 + a_1x_2 &\le h_1 + a_1h_2,\\ x_2 + bx_0 &\ge \ell_2 + b\ell_0, & x_0 - cx_1 &\le h_0 - c\ell_1,\\ & & x_2 - dx_1 &\le h_2 - d\ell_1. \end{aligned}$$

Put $w_i = h_i - \ell_i$ and $u_i = 1 - h_i$. Combining these inequalities at the six coordinate contacts gives

$$\begin{aligned} b\ell_0 &\le w_2 + \frac{d}{1+a_1d}w_1, & u_0 &\le \frac{a_0c}{a_0+c}w_1,\\ \ell_1 &\le \frac{bw_0 + w_2}{d+bc}, & u_1 &\le \frac{a_1(bw_0+w_2)}{1+a_0a_1b},\\ \ell_2 &\le bw_0 + \frac{ba_0c}{a_0+c}w_1, & u_2 &\le \frac{d}{1+a_1d}w_1. \end{aligned}$$

For example, at $x_1 = 0$ the two mixed inequalities bound x_0 and x_2 above; the lower inequality then gives the bound for ℓ_1. At $x_1 = 1$, the two upper inequalities and the lower inequality give the bound for u_1. The other four bounds follow by eliminating the remaining coordinate in the same way.

Set

$$Q = \frac{ba_0c}{a_0+c} + \frac{d}{1+a_1d}, \qquad V = \frac{1}{d+bc} + \frac{a_1}{1+a_0a_1b}.$$

Adding the bounds for ℓ_i and u_i and using $1 = w_i + \ell_i + u_i$ yields

$$1 \le w_0 + \frac{Q}{b}w_1 + \frac{1}{b}w_2, \qquad 1 \le w_1 + Vw_2 + bVw_0, \qquad 1 \le w_2 + bw_0 + Qw_1.$$

The identity

$$1 - QV = \frac{b(a_0a_1d - c)^2}{(a_0+c)(1+a_1d)(d+bc)(1+a_0a_1b)} \ge 0$$

controls the product of the two coefficients. Let $T = bw_0 + Qw_1 + w_2 > 0$. The first and third range bounds give $T \ge b, 1$, and the second gives

$$Q \le Qw_1 + QV(w_2 + bw_0) \le T.$$

It follows that $T = bw_0 + Qw_1 + w_2 \le T(w_0 + w_1 + w_2)$, proving the non-strict inequality.

Suppose equality holds. The sum

$$w_0(T-b) + w_1(T-Q) + w_2(T-1) = 0$$

has nonnegative terms and strictly positive weights. Therefore $b = Q = T = 1$. The middle range bound and $QV \le 1$ force $V = 1$, and the square identity gives $c = a_0a_1d$.

The bounds at $x_1 = 0, 1$ now read

$$\ell_1(d+c) \le w_0 + w_2, \qquad u_1(1 + a_0a_1) \le a_1(w_0 + w_2).$$

After division by their positive factors, their right sides sum to $V(w_0+w_2)=\ell_1+u_1$. Thus both are equalities. Since $c=a_0a_1d$, we obtain $u_1=a_1d\ell_1$. Write

$$X=h_0-c\ell_1, \qquad Z=h_2-d\ell_1.$$

The first equality gives $X+Z=\ell_0+\ell_2$. At $x_1=0$, the mixed inequalities give $x_0\le X$ and $x_2\le Z$. At $x_1=1$, the two upper inequalities give the same bounds, using $u_1=a_1d\ell_1$. In both cases the lower inequality gives $x_0+x_2\ge \ell_0+\ell_2=X+Z$. The two coordinate contacts are therefore $(X,0,Z)$ and $(X,1,Z)$. Their plane equations imply $A_1=0$, a contradiction. ☐

C.3 Moving a same-sign equality cover to the boundary

We now keep the endpoints of a given equality cover and ask whether its six same-sign pairs touch K at their corners. A window in the normalized range $[0,1]$ is *anchored* if its lower endpoint is zero or its upper endpoint is one. The motion below either gives all six original pair contacts or produces an anchored equality cover. We first need a two-window equality lemma.

Lemma 14 (An upper corner in a two-window equality cover)**.** *Let two affine coordinates X,Y on a convex set K have minimum zero and maximum one, both attained. Suppose the windows $[l,u]$ and $[m,v]$ cover K, where*

$$0<l<u\le 1, \qquad 0<m<v\le 1, \qquad (u-l)+(v-m)=1.$$

Then $(u,v)\in(X,Y)(K)$, unless $u=v=1$ and

$$X+Y\ge 1 \text{ on } K, \qquad (0,1),(1,0)\in(X,Y)(K).$$

Proof. First suppose $u,v<1$. Use the translated contacts from Lemma 2, with $w=u-l$ and $h=v-m$. Equality $w+h=1$ makes both Cauchy–Schwarz bounds in (4) equalities. All four distances a,b,c,d are positive, so equality also holds in the upper corner estimate: $bd=(w-q)(h-t)$. The contacts $R=(w+b,t)$ and $T=(q,h+d)$ are unchanged by clipping, because $b,d>0$. At parameter $\theta=b/(b+w-q)$, their segment has

$$x=w, \qquad y-h=\frac{bd-(w-q)(h-t)}{b+w-q}=0.$$

Here $q<w$ and $0<\theta<1$. Translating back gives $(u,v)\in(X,Y)(K)$.

It remains to treat $u=1$; the case $v=1$ is symmetric. Now $v-m=l$. The cap $X<l$ lies in $m\le Y\le v$. Applying Lemma 5 to its two excluded corners gives $t,s\ge 0$ with

$$m+tl\le Y+tX, \qquad Y-sX\le v-sl \quad \text{on } K.$$

When $v=1$, the second inequality holds with $s=0$. At contacts with $Y=0$, $Y=1$ and $X=0$ respectively,

$$m\le t(1-l), \qquad 1-v\le s(1-l), \qquad (t+s)l\le v-m=l.$$

Adding the first two gives $t+s\ge 1$, while the third gives $t+s\le 1$. Hence $s=1-t$, $m=(1-l)t$, and the supporting inequalities become

$$t(1-X)\le Y\le t+(1-t)X.$$

We have $t>0$ because $m>0$. If $t<1$, contacts with $Y=0$ and $Y=1$ both have $X=1$, and their segment contains $(1,v)$. If $t=1$, then $v=1$ and $X+Y\ge 1$. The contacts with $X=0$ and $Y=0$ give $(0,1)$ and $(1,0)$. ☐

Proposition 15. *Let K be compact and convex, satisfy (16), and attain zero and one in every coordinate. Suppose windows $0 < \ell_i < h_i < 1$ cover K, have total width one, and have all six same-sign pair caps empty. Then either*

$$K \cap \{x_i = \ell_i,\ x_j = \ell_j\} \neq \varnothing, \qquad K \cap \{x_i = h_i,\ x_j = h_j\} \neq \varnothing \quad (i \neq j),$$

or K has another cover with all six same-sign pair caps empty, total width one, all widths positive, and at least one anchored window.

Proof. Choose positive slopes a_i, b_i for the six supports in (8). Among windows with these six pair caps empty, the original cover minimizes total width: the six exclusions guarantee coverage, and (2), after rescaling the coordinates, gives a lower bound of one.

For the upper supports, set

$$N_i(a) = 1 - a_{i-1} + a_{i-1}a_{i+1}.$$

Change (h_i, h_{i+1}, h_{i-1}) in the direction $(1, a_{i+1}a_{i-1}, -a_{i-1})$. The right sides of the other two upper inequalities stay fixed, and the right side of upper inequality i increases at rate $1+\prod_k a_k > 0$. Lower endpoints remain fixed. A sufficiently small positive motion therefore preserves strictness and coverage. Its rate of change of total width is $N_i(a)$, so minimality gives $N_i(a) \geq 0$.

If K does not meet the pair corner $x_i = h_i,\ x_{i+1} = h_{i+1}$, the corresponding closed pair cap is empty. Indeed, positivity of its supporting coefficients forces every point of that closed cap to have both coordinates equal to their endpoints. Compactness then keeps the closed cap empty under small endpoint changes. Both signs of the motion are feasible, so $N_i(a) = 0$. Reflection gives the same conclusions for $N_i(b)$. Thus, if all six coefficients are positive, all six original pair corners meet K.

Otherwise, after cyclic relabelling and global reflection, assume $N_0(a) = 1 - a_2 + a_2a_1 = 0$. Keep the lower endpoints fixed and move the upper endpoints to

$$H = h + t(1, a_1a_2, -a_2), \qquad t = \min\left\{1 - h_0,\ \frac{1-h_1}{a_1a_2},\ \frac{h_2 - \ell_2}{a_2}\right\} > 0.$$

The supports remain valid and total width remains one. The new windows are ordered and contained in $[0,1]$, and either $H_0 = 1$, $H_1 = 1$, or $H_2 = \ell_2$.

We show that the third alternative is impossible. If $H_2 = \ell_2$, delete this zero-width window by the closedness argument in Lemma 3. Apply Lemma 14 to the two remaining windows. A point with $(x_0, x_1) = (H_0, H_1)$ contradicts the original empty upper pair cap, since $H_0 > h_0$ and $H_1 > h_1$.

In the exceptional case, $x_0 + x_1 \geq 1$ on K, and both $(0,1)$ and $(1,0)$ occur in its first two coordinates. At a contact with $x_2 = 1$, the plane equation gives

$$D - \lambda_2 = \lambda_0 x_0 + \lambda_1 x_1 \geq \min\{\lambda_0, \lambda_1\}.$$

At whichever of the two axis contacts has the smaller coefficient, the same equation forces $x_2 \geq 1$, hence $x_2 = 1$. This produces $(1,0,1)$ or $(0,1,1)$ in K, again contradicting an original strict upper pair cap. Thus the third width stays positive, including when several boundary conditions might be reached at once. An upper endpoint must reach one. Reflection treats a zero coefficient in the lower supports. □

The local motion and the missing-corner argument use no plane equation. The positive plane relation is needed to rule out the exceptional two-window limit and thereby keep all three widths positive.

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