A quantitative affine plank theorem for three planks in the plane

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Abstract

A plank is the region between two parallel hyperplanes. Its relative width with respect to a convex body is the ratio of their widths in the same direction. The Affine Plank Conjecture asserts that planks covering a convex body have total relative width at least 1. We prove the conjecture for three planks in the plane and propose a quantitative strengthening: in dimension d, planks of total relative width $R \leq 1$ leave at least $(1-R)^d$ of the body's volume uncovered. We prove this bound for three planks in the plane and for two planks in every dimension. For any number of planks, classical results give the bound for constant-width bodies and their affine images. Simplex examples show that the bound is sharp in every dimension. All results are formally verified in Lean 4. Companion material can be found at https://zazbrown.com/research/paper/affine-plank/quantitative-3planks-2d/

1 Introduction

A plank is the region between two parallel hyperplanes. Tarski's plank problem asks whether planks covering a convex body must have total width at least the body's minimum width. Bang proved this assertion [6]. At the end of the same paper he proposed its affine version: divide each plank's width by the width of the body in that plank's normal direction. Must the resulting relative widths add to at least 1? Unlike the comparison with minimum width, this formulation is unchanged by invertible affine transformations.

The Affine Plank Conjecture (APC) holds for centrally symmetric bodies, by Ball's theorem [5]; Akopyan and Polyanskii [1] give a recent short proof. The two-plank case goes back to Bang [7]; Gardner's relative-width measures give another approach [8], and Ambrus [2, §3] discusses its history. Hunter [10] studied equality for two planks and for a three-plank arrangement on a triangle. Verreault's survey [12] gives a broader account of plank problems and their applications.

For arbitrary convex bodies in $\mathbb{R}^d$ and arbitrary finite covers, Bakaev and Yehudayoff [4] proved the lower bound $2/(1+\sqrt{d})$ for the sum of relative widths. We prove the conjectured value 1 for three planks in the plane, with no restriction on their arrangement or the shape of the convex body (Theorem 2.1).

A natural quantitative question is how much of a convex body must remain uncovered when the planks have total relative width $R < 1$. We prove that at most three planks leave at least the fraction $(1-R)^2$ of the area of every planar convex body (Theorem 6.1). Planks parallel to the sides of a triangle attain the bound. We propose the Quantitative Affine Plank Conjecture (QAPC), which asserts that at least the fraction $(1-R)^d$ of the volume remains uncovered in dimension d. We prove it for two planks in every dimension.

Earlier partial-coverage results use total Euclidean width and the body's minimum width. They imply the proposed bound for constant-width bodies and their affine images. Section 10 states the general conjecture and explains these earlier results.

The proof has two stages. First, we prove the covering theorem by minimizing total relative width among coverings with fixed plank directions. Small motions of the plank boundaries reveal geometric constraints at a suitable minimum. In the plane, these constraints force $R \geq 1$.

The quantitative bound is equivalent to requiring that R plus the square root of the uncovered fraction is at least 1. We therefore minimize this sum. The covering theorem ensures that a minimum below 1 would leave positive area uncovered. Brunn–Minkowski (see [9]) and Borsuk–Ulam then give the geometric information needed for an area comparison: we enlarge the planks and compare the areas covered before and after. This comparison rules out a minimum below 1.

The minimization and region-counting arguments extend to higher dimensions. The remaining geometric steps use the fact that three planar plank normals are linearly dependent; they do not extend directly to three independent normals.

Sections 3–5 prove the covering theorem, including its supporting calculation. Section 6 then states the quantitative theorem; its proof ends in Section 9. Section 10 proves the two-plank bound in every dimension and records the constant-width case.

AI Declaration

GPT-6 Astra and Claude Fable 5.1 were used extensively to develop proof strategies, work out supporting arguments, write and simplify the Lean formalization, and draft and revise the manuscript. The author directed the project.

Mathematical software. The formalization uses Lean 4.35.0, Mathlib, and TauCeti. The proof terms are checked by Lean's kernel. The planar covering theorem, the planar quantitative theorem, and the two-plank theorem in every positive dimension also pass Lean's built-in Comparator against independently stated definitions and theorems.

2 The covering theorem

Let $K \subset \mathbb{R}^2$ be compact and convex with nonempty interior. Represent each plank by a nonconstant affine form $f_i : \mathbb{R}^2 \to \mathbb{R}$ and an interval $[a_i, b_i]$:

$$P_i = \{x : a_i \le f_i(x) \le b_i\}, \qquad a_i \le b_i.$$

The level lines of f_i are parallel to the plank's boundary lines. We denote their relative widths and total relative width by

$$r_i = \frac{b_i - a_i}{\max_K f_i - \min_K f_i}, \qquad R = \sum_i r_i.$$

The denominators are positive. Zero-width planks are allowed.

Theorem 2.1 (Three-plank planar APC)**.** *If three planks cover a compact convex planar body, their total relative width is at least one.*

We first prove the covering theorem, using no area inequality. Among coverings in fixed directions, minimize the sum of relative widths. Separation and linear algebra force at least six closed exterior contact cells at a suitable minimum. Helly's theorem and a short cube argument then give six supporting lines. Their contacts constrain the slopes, and the six-support calculation forces total relative width at least one. The proof, including the scalar calculation and parallel directions, is completed in Section 5.

3 A minimum among coverings

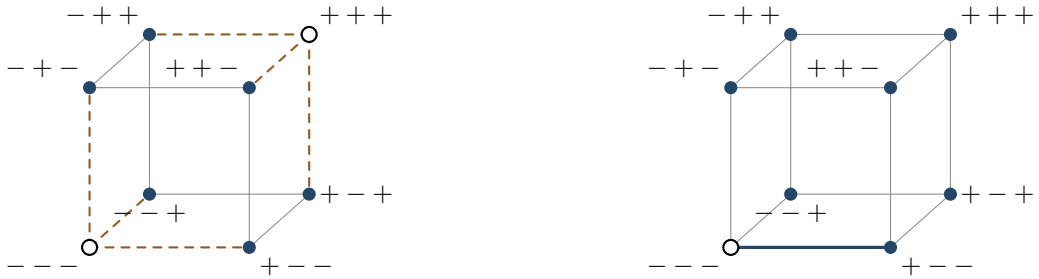
Shift and rescale each f_i so that $f_i(K) = [0, 1]$, and write $P_i = \{x : \ell_i \le f_i(x) \le h_i\}$. Clip its window $[\ell_i, h_i]$ to $[0, 1]$. Clipping preserves coverage and does not increase relative width. Write

$$e = (\ell_i, h_i)_{i=1}^m, \qquad r_i = h_i - \ell_i, \qquad R(e) = \sum_i r_i.$$

The admissible windows satisfy $0 \le \ell_i \le h_i \le 1$. For $s \in \{-,+\}^m$, the corresponding closed exterior cell is

$$C_s(e) = K \cap \bigcap_i \begin{cases} \{f_i \le \ell_i\}, & s_i = -, \\ \{f_i \ge h_i\}, & s_i = +. \end{cases}$$

Cells may be segments or points; the count concerns nonemptiness, not positive area. In a covering, every strict exterior cell is empty.



(a) Six cells at a cover minimum

(b) At least seven cells at a gap minimum

Figure 1: A filled vertex represents a nonempty closed exterior cell; a hollow vertex represents a missing cell. (a) At a cover minimum the six mixed cells remain. The dashed edges correspond to the six empty strict same-sign pair caps. (b) With at most one missing vertex, every edge has a present endpoint. Fixing two signs selects an edge, so every pair of windows has witnesses in all four exterior corners.

We use the known two-plank covering inequality [7]. It rules out a deficient three-plank cover on the boundary of the window box. A zero-width plank can be removed: its complement is dense in K, and the union of the remaining closed planks is closed. For a support-flush plank of width r, a homothetic copy of K about a point on the opposite support face, of ratio

$$R - r < \lambda < 1 - r,$$

strictly avoids that plank. The two remaining planks cover this copy with total relative width $(R - r)/\lambda < 1$, a contradiction. Such a λ exists whenever $R < 1$.

The covering configurations form a closed subset of the compact window box. If a cover with $R < 1$ exists, choose a minimum of R among covers; among these minima choose p with largest squared Euclidean endpoint norm. The preceding boundary argument makes every window strict: $0 < \ell_i < h_i < 1$.

Lemma 3.1 (Covering contact count)**.** *At an interior covering minimum chosen in this way, at least $2m$ closed exterior cells are nonempty.*

Proof. Retain the k cells nonempty at p. Every other closed cell stays empty under sufficiently small wall motions, by compactness of K. For each retained sign s, write its strict cuts at p as $g_i(x) > 0$, where $g_i = f_i - h_i$ for an upper sign and $g_i = \ell_i - f_i$ for a lower sign. Their affine image is convex and misses the open positive orthant. Separation gives coefficients $\alpha_i \ge 0$, not all zero, such that

$$\sum_i \alpha_i g_i(x) \le 0 \qquad (x \in K).$$

Keep this inequality valid by freezing its endpoint term:

$$L_s(e) = \sum_{s_i=+} \alpha_i h_i - \sum_{s_i=-} \alpha_i \ell_i = L_s(p).$$

The strict cell remains empty: a point satisfying every strict cut would make their nonzero nonnegative weighted sum positive.

If $k < 2m$, the k homogeneous equations $L_s(z) = 0$ in the $2m$ wall coordinates have a nonzero solution. For sufficiently small $t > 0$, both $p - tz$ and $p + tz$ remain in the window box and preserve coverage: the retained strict cells stay empty by separation, and the other cells stay empty by compactness. Linearity of R and minimality force both configurations to minimize R. But their average squared norm is $\|p\|^2 + t^2\|z\|^2 > \|p\|^2$, contradicting the choice of p. □

4 Covering certificates and six supports

We first treat the case of a positive affine relation; Section 5 reduces the general case to this one. Write S for the image of K in the three plank coordinates. Positive rescaling puts this image in a plane $x_0 + x_1 + x_2 = D$, with attained ranges $0 \le x_i \le D_i$. The windows are strict: $0 < \ell_i < h_i < D_i$. The functions x_i are the earlier f_i after positive rescaling; their ranges D_i need no longer be one. We keep the same window notation, and write $\delta_i = h_i - \ell_i$ for their coordinate widths. Thus $r_i = \delta_i / D_i$ and $R = \sum_i r_i$.

4.1 Empty pair caps and contact cells

An exterior sign chooses $x_i < \ell_i$ or $x_i > h_i$ for each coordinate. The eight signs are the vertices of a cube. Fixing two signs gives an edge; its two endpoints differ in the remaining coordinate. The closed exterior cells use weak inequalities. Figure 1 compares the cells at a cover minimum and a gap minimum.

Lemma 4.1 (Pair certificates)**.** *If the three windows cover S, every mixed exterior sign has an empty incident pair cap. Every empty pair cap has a positive supporting row on S.*

Proof. Let H_0, H_1, H_2 be the three strict exterior halfplanes of a mixed sign. Their intersection on the plane is nonempty: send the two equally signed coordinates far in their indicated direction and solve for the third. If every pair cap met S, every three of the four convex sets S, H_0, H_1, H_2 would intersect. Planar Helly would give a point in their common intersection, contradicting coverage.

For the supporting row, write the two strict cuts as $g > 0$ and $h > 0$. The convex image $\{(g(x), h(x)) : x \in S\}$ misses the open positive quadrant. Separation gives a nonzero pair $(\alpha, \beta) \ge 0$ with $\alpha g + \beta h \le 0$ on S: nonnegativity follows by letting either quadrant coordinate grow, and the zero bound by approaching its vertex. Each cut separately meets S, by the strict windows and attained ranges, so neither coefficient can vanish. Divide by $\alpha > 0$ to obtain $g + ah \le 0$ with $a = \beta/\alpha > 0$. □

Lemma 4.2 (Six contact cells)**.** *If the strict windows cover S and at least six closed exterior cells are nonempty, those cells are exactly the six mixed cells, and all six same-sign pair caps are empty.*

Proof. Let E be the cube vertices with nonempty closed cells. Both constant vertices cannot belong to E, since their points would give $D \le \sum_i \ell_i < \sum_i h_i \le D$.

An empty pair cap has at most one endpoint in E. Indeed, its positive supporting row forces both coordinates of any weak pair witness to equal their window walls. The plane equation then fixes the third coordinate, which cannot lie on both sides of its strict window. By Lemma 4.1, every mixed vertex in E therefore has a neighbor outside E.

At most two vertices are missing. If a constant vertex belonged to E, the opposite constant would be missing and at least five mixed vertices would need missing neighbors. One missing vertex has only three neighbors (Figure 1(a)); a constant and a nonopposite vertex have at most four neighbors outside that pair. Both possibilities are impossible. Thus E consists of the six mixed vertices. Each has only one neighbor outside E, its constant neighbor. Its empty incident pair must be the edge to that neighbor. These are precisely the six same-sign pairs. □

4.2 The six-support width inequality

The six supporting lines can be viewed as the sides of a containing hexagon, with redundant sides allowed. Each window leaves a lower and an upper margin in the corresponding coordinate range of S. The supporting rows bound these margins by the widths of the other two windows. A scalar comparison adds these three bounds; its full calculation is Lemma 4.4 in Section 4.3 below.

The geometric point is that the slopes in that comparison are not arbitrary. Two contact points on window walls force exactly the slope conditions it needs. This is the short argument below. Indices are cyclic, with $j = i + 1$ and $k = i + 2$.

Lemma 4.3 (Width from the contacts)**.** *Suppose all six same-sign pair caps of S are empty and all six mixed closed exterior cells are nonempty. Then the strict windows have total relative width at least one.*

Proof. Choose positive upper and lower supporting slopes a_i, b_i from Lemma 4.1, so that on S

$$x_i + a_i x_j \le h_i + a_i h_j, \qquad x_i + b_i x_j \ge \ell_i + b_i \ell_j.$$

Each row forces every weak same-sign pair witness onto both window walls.

In the order (i, j, k), the mixed cell $(-, +, +)$ supplies P: the positive upper row for (j, k) forces both coordinates onto their window walls. Similarly the cell $(-, -, +)$ supplies Q on the two lower walls. Thus there are points $P, Q \in S$ with $P_j = h_j$, $P_k = h_k$ and $Q_i = \ell_i$, $Q_j = \ell_j$. Set

$$\delta_j = h_j - \ell_j > 0, \qquad u = P_i - Q_i, \qquad v = Q_k - P_k.$$

The upper row for (j, k) at Q and the lower row for (i, j) at P give $a_j v \le \delta_j$ and $-b_i \delta_j \le u$. The plane gives $u + \delta_j = v$, whence

$$a_j(1 - b_i)\delta_j \le a_j(u + \delta_j) = a_j v \le \delta_j.$$

Cancel δ_j. Reflection proves $b_j(1 - a_i) \le 1$. Lemma 4.4 now gives the width bound. □

4.3 The numerical six-support calculation

This is the scalar comparison used in the cover-minimum argument. The first half of the proof bounds the two margins of each window by alternating three supporting rows. The second half adds those bounds: a cubic remainder has nonnegative pair coefficients and a nonnegative cycle coefficient. The box inequality below proves the latter sign. No plane equation or convexity is needed for this calculation. Indices are cyclic, with $j = i + 1$ and $k = i + 2$.

Lemma 4.4 (Numerical support criterion)**.** *Suppose a set of points meets both $x_i = 0$ and $x_i = D_i > 0$ for each i. Let $\ell_i \le h_i$ and let $a_i, b_i > 0$ satisfy, at every point,*

$$x_i + a_i x_j \le h_i + a_i h_j, \qquad x_i + b_i x_j \ge \ell_i + b_i \ell_j.$$

If $a_j(1 - b_i) \le 1$ and $b_j(1 - a_i) \le 1$ for every i, then $\sum_i (h_i - \ell_i)/D_i \ge 1$.

Proof. Put $\delta_i = h_i - \ell_i$, $L_i = 1 + b_i a_j b_k$ and $U_i = 1 + a_i b_j a_k$. To bound ℓ_i, reverse the lower row for (i, j), multiply the upper row for (j, k) by b_i, and reverse the lower row for (k, i) after multiplying it by $b_i a_j$. Adding these three inequalities cancels x_j, x_k and gives

$$L_i(\ell_i - x_i) \le b_i(\delta_j + a_j \delta_k).$$

Evaluate at $x_i = 0$; reflection gives the corresponding bound at $x_i = D_i$:

$$L_i \ell_i \le b_i(\delta_j + a_j \delta_k), \qquad U_i(D_i - h_i) \le a_i(\delta_j + b_j \delta_k). \tag{1}$$

Put $E_i = L_i U_i > 0$. Multiply these bounds by U_i, L_i and add $E_i\delta_i$:

$$S_i := E_i\delta_i + b_i U_i(\delta_j + a_j\delta_k) + a_i L_i(\delta_j + b_j\delta_k) \geq E_i D_i > 0.$$

Write $S_i = E_i\delta_i + F_i\delta_j + J_i\delta_k$, where

$$F_i = b_i U_i + a_i L_i, \qquad J_i = b_i a_j U_i + a_i b_j L_i.$$

We use the elementary box inequality

$$\prod_i u_i \leq 1 + \prod_i v_i + \sum_i u_i v_k \qquad (0 \leq u_i \leq 1 + v_i, \quad v_i \geq 0).$$

Indeed, the difference is affine in each u_i separately, so its minimum on the box is attained at a vertex. If any $u_i = 0$ it is nonnegative; at the remaining vertex, $u_i = 1 + v_i$ for all i, it is zero. Apply this with $u_i = a_i$, $v_i = a_i b_k$, and then interchange a, b. The slope assumptions give the required bounds on u_i; adding yields

$$H := 2\left(\prod_i a_i - 1\right)\left(\prod_i b_i - 1\right) + \prod_i (a_i + b_i) \geq 0.$$

Set $\eta = (\prod_i a_i)(\prod_i b_i) - 1$, the discrepancy between the product of all six slopes and one. The same square η^2 occurs in each of the following coefficient identities, obtained by direct multiplication:

$$E_i E_j - F_i J_j = \eta^2, \qquad 2\prod_i E_i - \prod_i F_i - \prod_i J_i = \eta^2 H.$$

Expanding the cubic and collecting these coefficients gives

$$\sum_i E_i\delta_i S_j S_k - \prod_i S_i = \eta^2\left[\sum_i (S_i - E_i\delta_i)\delta_j\delta_k + H\prod_i \delta_i\right] \geq 0. \tag{2}$$

Dividing by $\prod_i S_i > 0$ and using $S_i \geq E_i D_i$ gives $1 \leq \sum_i E_i\delta_i/S_i \leq \sum_i \delta_i/D_i$. □

5 Completion of the covering theorem

Proof of Theorem 2.1. Suppose a cover has $R < 1$. Normalize the forms. Their linear parts are dependent; reflecting some forms, together with their windows, gives a nonzero relation $\sum_i A_i f_i = c$ with $A_i \geq 0$. If necessary, relabel so $A_0 > 0$ and replace f_0 by $g_0 = f_0 - \varepsilon(f_1 + f_2)$ for small $\varepsilon > 0$. Then

$$A_0 g_0 + (A_1 + \varepsilon A_0) f_1 + (A_2 + \varepsilon A_0) f_2 = c$$

has all coefficients positive. Widen the first window from $[\ell_0, h_0]$ to $[\ell_0 - 2\varepsilon, h_0]$. Since $0 \leq f_1 + f_2 \leq 2$ on K, every formerly covered point remains covered. The new directional range is at least $1 - 2\varepsilon$, and

$$R_{\text{new}} \leq \frac{R + 2\varepsilon}{1 - 2\varepsilon} < 1$$

for sufficiently small ε. Normalize and clip again. The positive relation persists, with coefficients multiplied by positive directional ranges. This also handles parallel directions.

Keep these forms fixed and select the covering minimum of Section 3. It has strict windows, total relative width below one, and at least six closed exterior cells by Lemma 3.1. Positively rescale the coordinates to put their relation in the form $x_0 + x_1 + x_2 = D$. Lemma 4.2 gives the six mixed cells and six empty same-sign pair caps. Lemma 4.3, including the scalar calculation above, forces $R \geq 1$, a contradiction. □

Thus the covering theorem is complete before any area estimate is used. We turn to the quantitative extension.

6 The quantitative theorem

Write $|A|$ for the area of a measurable set $A \subset \mathbb{R}^2$, and put $V = |K| > 0$. For a family of planks, write G for its uncovered area and $R = \sum_i r_i$ for its total relative width. Put $t_+ = \max\{0, t\}$.

Theorem 6.1 (Three-plank planar QAPC)**.** *If $0 \le m \le 3$ and $P_1, \dots, P_m$ have relative widths $r_1, \dots, r_m$ with respect to K, then*

$$\boxed{\left| K \setminus \bigcup_{i=1}^{m} P_i \right| \ge |K| \left(1 - \sum_{i=1}^{m} r_i\right)_+^2 .} \tag{3}$$

The coefficient is sharp.

For sharpness, take a triangle with barycentric coordinates $\lambda_1, \lambda_2, \lambda_3$. The support-flush planks $0 \le \lambda_i \le r_i$, with $\sum_i r_i \le 1$, leave a triangle of linear scale $1 - \sum_i r_i$. For fewer planks set the unused widths to zero.

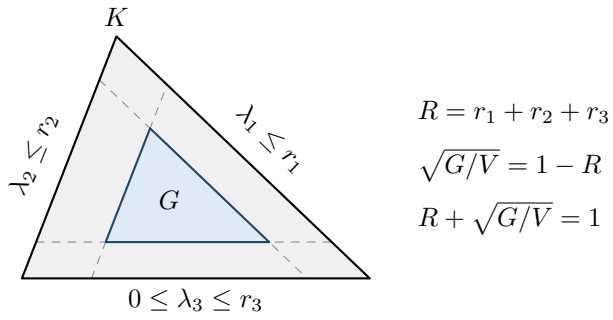


Figure 2: The sharp triangle example motivates the objective. The shaded central triangle has the same shape as K; its linear scale is the unused relative width. The three planks run along the sides of K.

In this example, $\sqrt{G/V} = 1 - R$ is the linear scale of the remaining triangle: it accounts exactly for the unused relative width. This motivates the quantity $C = R + \sqrt{G/V}$. Our aim is to show $C \ge 1$ for every configuration with $R \le 1$.

The one-plank area inequality alone does not give the sharp bound for several planks. For $R < 1$, it bounds the area covered by plank i by $(2r_i - r_i^2)V$. The union bound would give only

$$G \ge V \left(1 - 2R + \sum_i r_i^2\right),$$

whereas the desired lower bound is $V(1 - 2R + R^2)$. The difference is $2V \sum_{i<j} r_i r_j$, so the proof must control the overlap geometry.

A strict quantitative counterexample gives a minimum of C below one. The covering theorem already proved rules out $G = 0$. At the resulting positive-gap minimum, Brunn–Minkowski and Borsuk–Ulam force at least $2m + 1$ exterior cells. For two planks this is impossible: there are only four sign patterns. For three planks it gives seven of the eight patterns. These supply the pair centres needed for a single window-expansion argument, which gives the quantitative bound. We first establish the area comparisons used in that argument.

6.1 Four exterior corners

Lemma 6.2. *Let K be convex in a real vector space and let f, g be affine forms. If K meets all four weak exterior corner regions of two windows, then every pair of values in those windows is attained at a point of K.*

Proof. Put $F = (f, g)$ and let B be the rectangle of windows. Choose in $F(K)$ one witness from each exterior quadrant of B. Their convex hull contains B: otherwise a strict separator of a missing point would select a quadrant whose witness has at least that point's value, a contradiction. Thus $B \subset F(K)$, which is the required simultaneous attainment. □

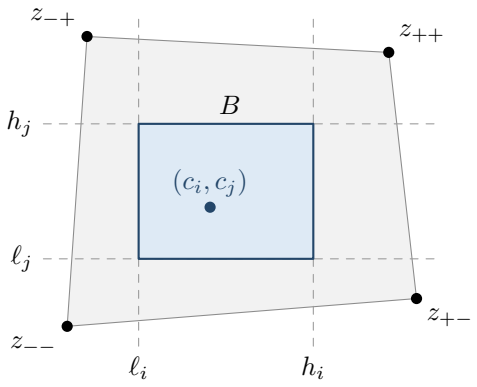


Figure 3: In the image under (f_i, f_j), one witness in each exterior corner forces their convex hull to contain the window rectangle B. In particular, the canonical pair of values (c_i, c_j) is attained in K.

6.2 Window scaling directly from the two caps

Keep a point c fixed in the normalized interval. Contracting a window toward c enlarges its two exterior caps. Brunn–Minkowski controls each cap separately, and adding their areas controls the window.

Lemma 6.3 (Planar window contraction)**.** *Let $K \subset \mathbb{R}^2$ be compact and convex, with area V. Let $f(K) = [0, 1]$, fix $c \in [0, 1]$, and put*

$$A(r) = |K \cap \{(1 - r)c \le f \le c + (1 - c)r\}|.$$

For $t, r \in [0, 1]$,

$$A(tr) \le t^2 A(r) + 2tr(1 - t)V. \tag{4}$$

Proof. The two cap areas at fractional depth s are

$$v_-(s) = |K \cap \{f \le sc\}|, \qquad v_+(s) = |K \cap \{f \ge 1 - s(1 - c)\}|.$$

Here depth one reaches the line $f = c$. For either cap write $v = v_-$ or $v = v_+$, and put $u = \sqrt{v(1)}$, $b = \sqrt{v(1 - r)}$. For the lower cap, a homothety of ratio $1 - r$ about a support point with $f = 0$ takes the depth-one cap into the depth-$(1 - r)$ cap; its square-root area is multiplied by $1 - r$. Moreover, a convex combination $(1 - t)x + ty$ of points in these two caps lies in K and has $f \le (1 - tr)c$. Reflection gives the upper-cap versions. Area scaling and Brunn–Minkowski therefore give

$$b \ge (1 - r)u, \qquad \sqrt{v(1 - tr)} \ge (1 - t)u + tb.$$

Squaring the second inequality and applying the first to its cross term yields

$$v(1 - tr) \ge t^2 v(1 - r) + (1 - t^2 - 2tr(1 - t))v(1).$$

Sum over the two caps. Their anchor areas sum to V, and their areas at depth $1 - r$ sum to $V - A(r)$, since boundary lines have zero area. Rearranging gives (4). □

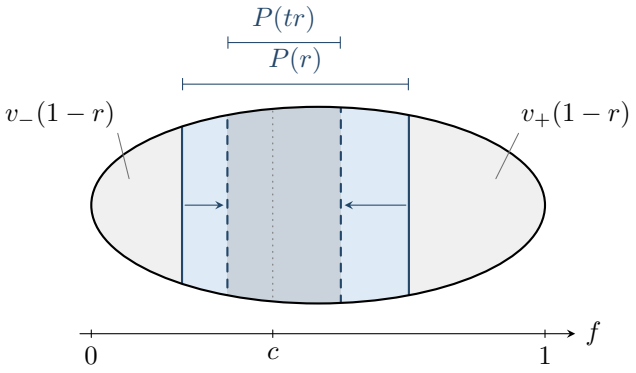


Figure 4: Shrinking the window from width r to tr moves its walls toward $f = c$ and enlarges both exterior caps. At depth one the caps meet at $f = c$, so $v_-(1) + v_+(1) = V$. At depth $1 - r$, their total area is $V - A(r)$.

6.3 Expand once to total width one

This estimate does not depend on the number of planks.

Proposition 6.4 (Finite-family estimate)**.** *Let K be a compact convex planar set of area V, and let $P_i = \{\ell_i \le f_i \le h_i\}$, $1 \le i \le m$, where the affine forms have $f_i(K) = [0, 1]$, and $0 \le \ell_i \le h_i \le 1$. Put $r_i = h_i - \ell_i$ and suppose $0 < R = \sum_i r_i < 1$. For each pair $i < j$, suppose there is $z_{ij} \in K$ with $f_i(z_{ij}) = c_i$ and $f_j(z_{ij}) = c_j$, where $c_i = \ell_i/(1 - r_i)$. If every intersection of three distinct planks has area zero, then the actual uncovered area satisfies*

$$G \ge V(1 - R)^2.$$

Since $c_i \in [\ell_i, h_i]$, four exterior corners for each pair supply the required centres by Lemma 6.2.

Proof. Expand the canonical windows to relative widths $\widehat{r}_i = r_i/R$. Write $A_i = |K \cap P_i|$, $B_{ij} = |K \cap P_i \cap P_j|$, and use hats for the expanded quantities. For each pair, the homothety $x \mapsto (1 - R)z_{ij} + Rx$ stays in K and satisfies

$$f_i\big((1 - R)z_{ij} + Rx\big) = (1 - R)c_i + Rf_i(x),$$

and likewise for j. It maps each expanded window onto the original window, hence takes the expanded pair intersection into the original one. Area scaling therefore gives $R^2\widehat{B}_{ij} \le B_{ij}$. Apply window contraction with $t = R$ and sum over i using $\sum_i \widehat{r}_i = 1$. Bonferroni's inequality for the expanded planks inside K gives the second inequality below:

$$\sum_i A_i \le R^2 \sum_i \widehat{A}_i + 2R(1 - R)V,$$
$$\sum_i \widehat{A}_i \le V + \sum_{i<j} \widehat{B}_{ij}.$$

The null triple intersections give $G = V - \sum_i A_i + \sum_{i<j} B_{ij}$. Substitution and the pair inequalities yield

$$G \ge V - R^2V - 2R(1 - R)V = (1 - R)^2V.$$

□

7 A minimum with positive uncovered area

For $m = 2, 3$, assume the quantitative bound for $m - 1$ planks. The one-plank case is Lemma 6.3 with $r = 1$.

Normalize each affine form f_i to have range $[0,1]$ on K and clip its window to that interval. Clipping preserves gap area and does not increase the relative budget; a disjoint window can be discarded. Put

$$e = (\ell_i, h_i)_{i=1}^m, \qquad r_i = h_i - \ell_i, \qquad R(e) = \sum_i r_i, \qquad V = |K|,$$

and let $G(e)$ denote the actual uncovered area. It is continuous on

$$D = \{e : 0 \le \ell_i \le h_i \le 1,\ R(e) \le 1\}.$$

The target is zero when the original budget is at least one.

On ∂D the bound for $m-1$ planks gives $G \ge V(1-R)^2$. A zero-width plank is null. For a support-flush plank of width $r < 1$, contract K by $1-r$ toward a point on the opposite support face. This copy avoids that plank outside a null boundary line, has area $(1-r)^2 V$, and multiplies all directional widths by $1-r$. Applying the bound for $m-1$ planks to the remaining planks gives

$$(1-r)^2 V \left(1 - \frac{R-r}{1-r}\right)^2 = V(1-R)^2.$$

The faces $R = 1$ and $r = 1$ have zero target.

A strict counterexample therefore makes $C(e) = R(e) + \sqrt{G(e)/V}$ attain a minimum $B < 1$. Its nonempty compact minimum set $Z = \{e \in D : C(e) = B\}$ lies in $\operatorname{int} D$. Choose $p \in Z$ maximizing the Euclidean squared norm of its endpoint coordinates. This choice prevents two symmetric perturbations $p - z$ and $p + z$, with $z \neq 0$, from both being minimizers: their mean squared norm is larger than the squared norm of p. We will use this to count the exterior cells. The covering theorem proved above, together with the known two-plank case, now gives $G(p) > 0$. Indeed, if $G(p) = 0$, the closed planks cover K: any missed point has a neighborhood outside them containing positive body area. Coverage would force $R(p) \ge 1$, contrary to $C(p) < 1$.

The remaining cell count uses the positive gap to normalize one square-root cell-area coordinate. Borsuk–Ulam finds opposite wall motions with proportional area vectors if there are too few cells. Brunn–Minkowski then forces both configurations to minimize C, contradicting the choice of p.

Lemma 7.1. *Let $m = 2$ or $m = 3$, and suppose the quantitative bound holds for $m-1$ planks. A strict m-plank counterexample has an interior minimum p of C with positive uncovered area and at least $2m+1$ nonempty closed exterior cells.*

Proof. Use p, B and Z from the minimum construction above. Retain the k cells nonempty at p, including segments and singletons. Every other cell stays empty near p, by compactness of K. On this neighborhood put

$$b(e) = (\sqrt{|C_s(e)|/V})_{s=1}^k, \qquad C(e) = R(e) + |b(e)|.$$

Here $|b(e)|$ is the Euclidean norm, equal to $\sqrt{G(e)/V}$. The map b is continuous: moving the endpoints changes cell indicators pointwise outside finitely many null boundary lines, so dominated convergence applies. Choose a small sphere of perturbations z around zero, with both $p \pm z$ in this neighborhood and in D.

We claim that its opposite vectors $b(p-z)$ and $b(p+z)$ cannot be positive scalar multiples. If they were, their zero coordinates would agree. For a positive coordinate, Brunn–Minkowski and $\frac{1}{2}C_s(p-z) + \frac{1}{2}C_s(p+z) \subseteq C_s(p)$ give

$$b_s(p-z) + b_s(p+z) \le 2b_s(p).$$

This also holds when both left-hand terms vanish. Nonnegativity and proportionality therefore imply

$$|b(p-z)| + |b(p+z)| = |b(p-z) + b(p+z)| \le 2|b(p)|.$$

Since R is linear, $C(p-z)+C(p+z) \le 2C(p)$. Minimality forces both endpoints into Z, contradicting the maximal squared norm of p: the squared Euclidean norm is strictly convex, so its average at two distinct endpoints exceeds its value at their midpoint.

Borsuk–Ulam says that every continuous map $S^N \to \mathbb{R}^j$, with $j \le N$, agrees at some antipodal pair [11, Theorem 2.1.1]. Since $G(p) > 0$, choose a coordinate $b_a(p) > 0$ and shrink the sphere so $b_a(p \pm z) > 0$. If $k \le 2m$, apply Borsuk–Ulam to

$$z \longmapsto \left(\frac{b_s(p+z)}{b_a(p+z)}\right)_{s \ne a} \in \mathbb{R}^{k-1}.$$

Equal opposite ratios give positive proportional vectors, again contradicting the claim. Hence $k \ge 2m+1$. □

8 Two planks: five cells would be required

Proposition 8.1. *Two arbitrary planks of total relative width R in a compact convex planar body K leave area at least $|K|(1-R)_+^2$.*

Proof. A strict counterexample would give a minimum with at least five nonempty exterior cells by Lemma 7.1. Two planks have only four sign patterns, a contradiction. □

9 Three planks: seven cells give the area bound

Suppose a strict three-plank counterexample exists. The two-plank theorem supplies the boundary input for Lemma 7.1. Choose its interior minimum p, with $C(p) < 1$, positive gap, and at least seven of the eight exterior cells present. These supply all four weak exterior corners for every pair: every edge of the cube has a present endpoint (Figure 1(b)). Lemma 6.2 puts each pair's canonical centre inside K.

The strict triple intersection is empty. To see this, take any nonzero affine relation among the three forms and reflect signs so that its coefficients are nonnegative. A point strictly inside all three windows would exclude the all-lower and all-upper closed exterior cells: evaluating the relation at either cell would put its constant strictly below or strictly above its value at that interior point. This contradicts the presence of seven cells. The coefficients need not all be positive; at least one positive coefficient makes both inequalities strict. Thus no perturbation is needed in the quantitative step.

The closed triple intersection lies in the six boundary lines and has area zero. Proposition 6.4 gives $G \ge V(1-R)^2$, contradicting $C(p) < 1$ and completing Theorem 6.1.

10 The Quantitative Affine Plank Conjecture

The planar theorem suggests the following strengthening of the Affine Plank Conjecture. Here volume is d-dimensional Lebesgue measure.

Conjecture 10.1. *For every $d \ge 1$, every convex body $K \subset \mathbb{R}^d$, and every finite family of planks P_i whose relative widths with respect to K are r_i,*

$$\frac{|K \setminus \bigcup_i P_i|}{|K|} \ge \left(1 - \sum_i r_i\right)_+^d.$$

Support-flush planks on a simplex give equality: in barycentric coordinates $\lambda_0, \dots, \lambda_d$, remove the regions $\lambda_i \le r_i$ with $R = \sum_i r_i < 1$. The remaining simplex is obtained by writing

$\lambda_i = r_i + (1-R)\eta_i$ and has volume $(1-R)^d|K|$. A single support-flush plank already shows sharpness.

We are not aware of an earlier statement in this generality. The proposed strengthening is the direction-relative, affine-invariant budget R; the exponent and the use of homotheties already appear in earlier work. Write $W = \sum_i \mathrm{width}(P_i)$ for total Euclidean width and $D_{\min}(K)$ for the minimum Euclidean width of K. Bang's erosion and translated-cell argument [6, Lemma 1 and pp. 991–992], in the form of Bakaev–Polyanskii's [3, Corollary 7 and Lemma 8], implies

$$\frac{|K \setminus \bigcup_i P_i|}{|K|} \geq \left(1 - \frac{W}{D_{\min}(K)}\right)_+^d. \tag{5}$$

Indeed, those two results bound the gap volume below by the volume of an erosion containing a translate of $(1 - W/D_{\min}(K))K$ when $W < D_{\min}(K)$. This is a consequence of their machinery, rather than an assertion that Bang stated this formula. Since $R \leq W/D_{\min}(K)$, the affine budget asks for a larger lower bound in general.

10.1 Further cases

Proposition 10.2 (Two planks in every dimension)**.** *For every $d \geq 1$, two planks of total relative width R in a convex body $K \subset \mathbb{R}^d$ leave volume at least $|K|(1-R)_+^d$.*

Proof. Write G for the uncovered volume. We first establish the one-plank bound. Normalize its affine form f to range $[0,1]$ on K. After clipping, write the window as $[\ell, \ell + r]$ with $0 \leq \ell \leq \ell + r \leq 1$. The case $r = 1$ is immediate. Otherwise put $c = \ell/(1-r)$ and take support points $a_0, a_1 \in K$ with $f(a_j) = j$. The homotheties $H_j(x) = a_j + (1-r)(x - a_j)$ send K into itself by convexity. Moreover,

$$H_0(K \cap \{f \leq c\}) \subset K \cap \{f \leq \ell\}, \qquad H_1(K \cap \{f \geq c\}) \subset K \cap \{f \geq \ell + r\}.$$

The two source caps partition K up to a null hyperplane. Their images lie outside the window up to null boundary hyperplanes and have disjoint interiors, so $G \geq (1-r)^d|K|$. The simplex example above attains equality.

The minimum argument behind Lemma 7.1 extends to every positive dimension and every finite plank count. The ℓ^d norm of the normalized cell-volume roots is $(G/V)^{1/d}$, so the same objective becomes $C = R + (G/V)^{1/d}$. Assuming the $(m-1)$-plank bound, Brunn–Minkowski, antipodal matching and the same squared-norm tie-break select a counterexample minimum with at least $2m$ nonempty cells, and $2m+1$ if its gap is positive.

For two planks, the one-plank bound supplies the boundary input for this minimum argument. A strict counterexample would therefore give an interior minimum of $C = R + (G/V)^{1/d}$ below one. If $G = 0$, the closed planks cover K, and the two-plank covering theorem [7] forces $R \geq 1$. If $G > 0$, the cell count forces at least five nonempty exterior cells. There are only four sign patterns for two planks. Both alternatives are impossible. □

Constant-width bodies and their affine images. For a body of constant width D, every directional width equals D, so $R = W/D = W/D_{\min}(K)$. Thus (5) proves the conjecture for every finite plank family on such a body. Relative widths and uncovered-volume fractions are preserved by a simultaneous invertible affine change of the body and planks, giving the same conclusion for all its affine images.

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