A quantitative affine plank theorem for three planks in the plane

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26 October 2026

Abstract

A plank is the region between two parallel hyperplanes. Its relative width with respect to a convex body is the ratio of their widths in the same direction. The Affine Plank Conjecture asserts that planks covering a convex body have total relative width at least 1. We prove the conjecture for three planks in the plane and propose a quantitative strengthening: in dimension d, planks of total relative width $R \leq 1$ leave at least $(1-R)^d$ of the body's volume uncovered. We prove this bound for three planks in the plane and for two planks in every dimension. For any number of planks, classical results give the bound for constant-width bodies and their affine images. Simplex examples show that the bound is sharp in every dimension. All results are formally verified in Lean 4. Companion material can be found at https://zazbrown.com/research/paper/affine-plank/quantitative-3planks-2d/

1 Introduction

A plank is the region between two parallel hyperplanes. Tarski's plank problem asks whether planks covering a convex body must have total width at least the body's minimum width. Bang proved this assertion [8]. At the end of the same paper he proposed its affine version: divide each plank's width by the width of the body in that plank's normal direction. Must the resulting relative widths add to at least 1? Unlike the comparison with minimum width, this formulation is unchanged by invertible affine transformations.

The Affine Plank Conjecture (APC) holds for centrally symmetric bodies, by Ball's theorem [7]; Akopyan and Polyanskii [1] give a recent short proof. The two-plank case goes back to Bang [9]; Gardner's relative-width measures give another approach [11], and Ambrus [2, §3] discusses its history. Hunter [13] studied equality for two planks and for a three-plank arrangement on a triangle. Verreault's survey [15] gives a broader account of plank problems and their applications.

For arbitrary convex bodies in $\mathbb{R}^d$ and arbitrary finite covers, Bakaev and Yehudayoff [5] proved the lower bound $2/(1+\sqrt{d})$ for the sum of relative widths. We prove the conjectured value 1 for three planks in the plane, with no restriction on their arrangement or the shape of the convex body (Theorem 2.1). We also classify all three-plank coverings of total relative width 1, including parallel directions and zero-width planks (Appendix A).

A natural quantitative question is how much of a convex body must remain uncovered when the planks have total relative width $R < 1$. We prove that at most three planks leave at least the fraction $(1-R)^2$ of the area of every planar convex body (Theorem 6.1). Planks parallel to the sides of a triangle attain the bound. We propose the Quantitative Affine Plank Conjecture (QAPC), which asserts that at least the fraction $(1-R)^d$ of the volume remains uncovered in dimension d. We prove it for two planks in every dimension.

Earlier partial-coverage results use total Euclidean width and the body's minimum width. They imply the proposed bound for constant-width bodies and their affine images. Section 10 states the general conjecture and explains these earlier results. It also compares the conjecture with recent stability and annulus results.

The proof has two stages. First, we prove the covering theorem by minimizing total relative width among coverings with fixed plank directions. Small motions of the plank boundaries reveal geometric constraints at a suitable minimum. In the plane, these constraints force $R \geq 1$.

The quantitative bound is equivalent to requiring that R plus the square root of the uncovered fraction is at least 1. We therefore minimize this sum. The covering theorem ensures that a minimum below 1 would leave positive area uncovered. Brunn–Minkowski (see [12]) and Borsuk–Ulam then give the geometric information needed for an area comparison: we enlarge the planks and compare the areas covered before and after. This comparison rules out a minimum below 1.

The minimization and region-counting arguments extend to higher dimensions. The remaining geometric steps use the fact that three planar plank normals are linearly dependent; they do not extend directly to three independent normals.

Sections 3–5 prove the covering theorem, including its supporting calculation. Section 6 then states the quantitative theorem; its proof ends in Section 9. Section 10 proves the two-plank bound in every dimension and records the constant-width case. It also states finite-body reductions, proved in Appendix B.

AI Declaration

GPT-6 Astra and Claude Fable 5.1 were used extensively to develop proof strategies, work out supporting arguments, write and simplify the Lean formalization, and draft and revise the manuscript. The author directed the project.

Mathematical software. The formalization uses Lean 4.35.0, Mathlib, and TauCeti. The proof terms are checked by Lean's kernel. The planar covering theorem, the planar quantitative theorem, and the two-plank theorem in every positive dimension also pass Lean's built-in Comparator against independently stated definitions and theorems.

2 The covering theorem

Let $K \subset \mathbb{R}^2$ be compact and convex with nonempty interior. Represent each plank by a nonconstant affine form $f_i : \mathbb{R}^2 \to \mathbb{R}$ and an interval $[a_i, b_i]$:

$$P_i = \{x : a_i \leq f_i(x) \leq b_i\}, \qquad a_i \leq b_i.$$

The level lines of f_i are parallel to the plank's boundary lines. We denote their relative widths and total relative width by

$$r_i = \frac{b_i - a_i}{\max_K f_i - \min_K f_i}, \qquad R = \sum_i r_i.$$

The denominators are positive. Zero-width planks are allowed.

Theorem 2.1 (Three-plank planar APC)**.** *If three planks cover a compact convex planar body, their total relative width is at least one.*

We first prove the covering theorem, using no area inequality. Among coverings in fixed directions, minimize the sum of relative widths. Separation and linear algebra force at least six closed exterior contact cells at a suitable minimum. Helly's theorem and a short cube argument then give six supporting lines. Their contacts constrain the slopes, and the six-support calculation forces total relative width at least one. The proof, including the scalar calculation and parallel directions, is completed in Section 5.

3 A minimum among coverings

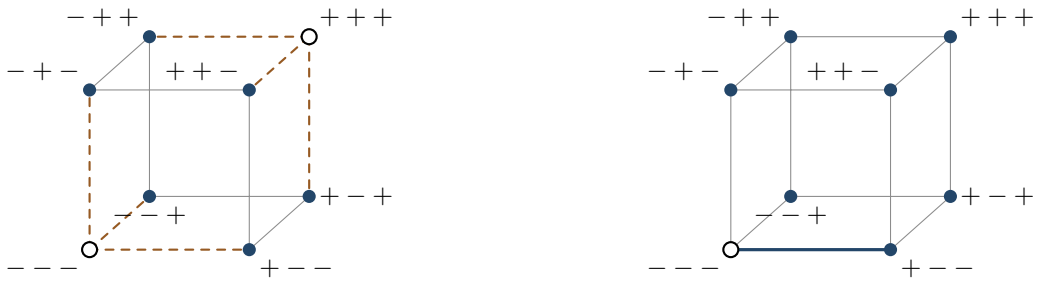
Shift and rescale each f_i so that $f_i(K) = [0,1]$, and write $P_i = \{x : \ell_i \le f_i(x) \le h_i\}$. Clip its window $[\ell_i, h_i]$ to $[0,1]$. Clipping preserves coverage and does not increase relative width. Write

$$e = (\ell_i, h_i)_{i=1}^m, \qquad r_i = h_i - \ell_i, \qquad R(e) = \sum_i r_i.$$

The admissible windows satisfy $0 \le \ell_i \le h_i \le 1$. For $s \in \{-,+\}^m$, the corresponding closed exterior cell is

$$C_s(e) = K \cap \bigcap_i \begin{cases} \{f_i \le \ell_i\}, & s_i = -, \\ \{f_i \ge h_i\}, & s_i = +. \end{cases}$$

Cells may be segments or points; the count concerns nonemptiness, not positive area. In a covering, every strict exterior cell is empty.



(a) Six cells at a cover minimum

(b) At least seven cells at a gap minimum

Figure 1: A filled vertex represents a nonempty closed exterior cell; a hollow vertex represents a missing cell. (a) At a cover minimum the six mixed cells remain. The dashed edges correspond to the six empty strict same-sign pair caps. (b) With at most one missing vertex, every edge has a present endpoint. Fixing two signs selects an edge, so every pair of windows has witnesses in all four exterior corners.

We use the known two-plank covering inequality [9]. It rules out a deficient three-plank cover on the boundary of the window box. A zero-width plank can be removed: its complement is dense in K, and the union of the remaining closed planks is closed. For a support-flush plank of width r, a homothetic copy of K about a point on the opposite support face, of ratio

$$R - r < \lambda < 1 - r,$$

strictly avoids that plank. The two remaining planks cover this copy with total relative width $(R-r)/\lambda < 1$, a contradiction. Such a λ exists whenever $R < 1$.

The covering configurations form a closed subset of the compact window box. If a cover with $R < 1$ exists, choose a minimum of R among covers; among these minima choose p with largest squared Euclidean endpoint norm. The preceding boundary argument makes every window strict: $0 < \ell_i < h_i < 1$.

Lemma 3.1 (Covering contact count)**.** *At an interior covering minimum chosen in this way, at least $2m$ closed exterior cells are nonempty.*

Proof. Retain the k cells nonempty at p. Every other closed cell stays empty under sufficiently small wall motions, by compactness of K. For each retained sign s, write its strict cuts at p as $g_i(x) > 0$, where $g_i = f_i - h_i$ for an upper sign and $g_i = \ell_i - f_i$ for a lower sign. Their affine image is convex and misses the open positive orthant. Separation gives coefficients $\alpha_i \ge 0$, not all zero, such that

$$\sum_i \alpha_i g_i(x) \le 0 \qquad (x \in K).$$

Keep this inequality valid by freezing its endpoint term:

$$L_s(e) = \sum_{s_i=+} \alpha_i h_i - \sum_{s_i=-} \alpha_i \ell_i = L_s(p).$$

The strict cell remains empty: a point satisfying every strict cut would make their nonzero nonnegative weighted sum positive.

If $k < 2m$, the k homogeneous equations $L_s(z) = 0$ in the $2m$ wall coordinates have a nonzero solution. For sufficiently small $t > 0$, both $p - tz$ and $p + tz$ remain in the window box and preserve coverage: the retained strict cells stay empty by separation, and the other cells stay empty by compactness. Linearity of R and minimality force both configurations to minimize R. But their average squared norm is $\|p\|^2 + t^2\|z\|^2 > \|p\|^2$, contradicting the choice of p. □

4 Covering certificates and six supports

We first treat the case of a positive affine relation; Section 5 reduces the general case to this one. Write S for the image of K in the three plank coordinates. Positive rescaling puts this image in a plane $x_0 + x_1 + x_2 = D$, with attained ranges $0 \le x_i \le D_i$. The windows are strict: $0 < \ell_i < h_i < D_i$. The functions x_i are the earlier f_i after positive rescaling; their ranges D_i need no longer be one. We keep the same window notation, and write $\delta_i = h_i - \ell_i$ for their coordinate widths. Thus $r_i = \delta_i / D_i$ and $R = \sum_i r_i$.

4.1 Empty pair caps and contact cells

An exterior sign chooses $x_i < \ell_i$ or $x_i > h_i$ for each coordinate. The eight signs are the vertices of a cube. Fixing two signs gives an edge; its two endpoints differ in the remaining coordinate. The closed exterior cells use weak inequalities. Figure 1 compares the cells at a cover minimum and a gap minimum.

Lemma 4.1 (Pair certificates)**.** *If the three windows cover S, every mixed exterior sign has an empty incident pair cap. Every empty pair cap has a positive supporting row on S.*

Proof. Let H_0, H_1, H_2 be the three strict exterior halfplanes of a mixed sign. Their intersection on the plane is nonempty: send the two equally signed coordinates far in their indicated direction and solve for the third. If every pair cap met S, every three of the four convex sets S, H_0, H_1, H_2 would intersect. Planar Helly would give a point in their common intersection, contradicting coverage.

For the supporting row, write the two strict cuts as $g > 0$ and $h > 0$. The convex image $\{(g(x), h(x)) : x \in S\}$ misses the open positive quadrant. Separation gives a nonzero pair $(\alpha, \beta) \ge 0$ with $\alpha g + \beta h \le 0$ on S: nonnegativity follows by letting either quadrant coordinate grow, and the zero bound by approaching its vertex. Each cut separately meets S, by the strict windows and attained ranges, so neither coefficient can vanish. Divide by $\alpha > 0$ to obtain $g + ah \le 0$ with $a = \beta/\alpha > 0$. □

Lemma 4.2 (Six contact cells)**.** *If the strict windows cover S and at least six closed exterior cells are nonempty, those cells are exactly the six mixed cells, and all six same-sign pair caps are empty.*

Proof. Let E be the cube vertices with nonempty closed cells. Both constant vertices cannot belong to E, since their points would give $D \le \sum_i \ell_i < \sum_i h_i \le D$.

An empty pair cap has at most one endpoint in E. Indeed, its positive supporting row forces both coordinates of any weak pair witness to equal their window walls. The plane equation then fixes the third coordinate, which cannot lie on both sides of its strict window. By Lemma 4.1, every mixed vertex in E therefore has a neighbor outside E.

At most two vertices are missing. If a constant vertex belonged to E, the opposite constant would be missing and at least five mixed vertices would need missing neighbors. One missing vertex has only three neighbors (Figure 1(a)); a constant and a nonopposite vertex have at most four neighbors outside that pair. Both possibilities are impossible. Thus E consists of the six mixed vertices. Each has only one neighbor outside E, its constant neighbor. Its empty incident pair must be the edge to that neighbor. These are precisely the six same-sign pairs. □

4.2 The six-support width inequality

The six supporting lines can be viewed as the sides of a containing hexagon, with redundant sides allowed. Each window leaves a lower and an upper margin in the corresponding coordinate range of S. The supporting rows bound these margins by the widths of the other two windows. A scalar comparison adds these three bounds; its full calculation is Lemma 4.4 in Section 4.3 below.

The geometric point is that the slopes in that comparison are not arbitrary. Two contact points on window walls force exactly the slope conditions it needs. This is the short argument below. Indices are cyclic, with $j = i + 1$ and $k = i + 2$.

Lemma 4.3 (Width from the contacts)**.** *Suppose all six same-sign pair caps of S are empty and all six mixed closed exterior cells are nonempty. Then the strict windows have total relative width at least one.*

Proof. Choose positive upper and lower supporting slopes a_i, b_i from Lemma 4.1, so that on S

$$x_i + a_i x_j \le h_i + a_i h_j, \qquad x_i + b_i x_j \ge \ell_i + b_i \ell_j.$$

Each row forces every weak same-sign pair witness onto both window walls.

In the order (i, j, k), the mixed cell $(-, +, +)$ supplies P: the positive upper row for (j, k) forces both coordinates onto their window walls. Similarly the cell $(-, -, +)$ supplies Q on the two lower walls. Thus there are points $P, Q \in S$ with $P_j = h_j$, $P_k = h_k$ and $Q_i = \ell_i$, $Q_j = \ell_j$. Set

$$\delta_j = h_j - \ell_j > 0, \qquad u = P_i - Q_i, \qquad v = Q_k - P_k.$$

The upper row for (j, k) at Q and the lower row for (i, j) at P give $a_j v \le \delta_j$ and $-b_i \delta_j \le u$. The plane gives $u + \delta_j = v$, whence

$$a_j(1 - b_i)\delta_j \le a_j(u + \delta_j) = a_j v \le \delta_j.$$

Cancel δ_j. Reflection proves $b_j(1 - a_i) \le 1$. Lemma 4.4 now gives the width bound. □

4.3 The numerical six-support calculation

This is the scalar comparison used in the cover-minimum argument. The first half of the proof bounds the two margins of each window by alternating three supporting rows. The second half adds those bounds: a cubic remainder has nonnegative pair coefficients and a nonnegative cycle coefficient. The box inequality below proves the latter sign. No plane equation or convexity is needed for this calculation. Indices are cyclic, with $j = i + 1$ and $k = i + 2$.

Lemma 4.4 (Numerical support criterion)**.** *Suppose a set of points meets both $x_i = 0$ and $x_i = D_i > 0$ for each i. Let $\ell_i \le h_i$ and let $a_i, b_i > 0$ satisfy, at every point,*

$$x_i + a_i x_j \le h_i + a_i h_j, \qquad x_i + b_i x_j \ge \ell_i + b_i \ell_j.$$

If $a_j(1 - b_i) \le 1$ and $b_j(1 - a_i) \le 1$ for every i, then $\sum_i (h_i - \ell_i)/D_i \ge 1$.

Proof. Put $\delta_i = h_i - \ell_i$, $L_i = 1 + b_i a_j b_k$ and $U_i = 1 + a_i b_j a_k$. To bound ℓ_i, reverse the lower row for (i,j), multiply the upper row for (j,k) by b_i, and reverse the lower row for (k,i) after multiplying it by $b_i a_j$. Adding these three inequalities cancels x_j, x_k and gives

$$L_i(\ell_i - x_i) \le b_i(\delta_j + a_j\delta_k).$$

Evaluate at $x_i = 0$; reflection gives the corresponding bound at $x_i = D_i$:

$$L_i\ell_i \le b_i(\delta_j + a_j\delta_k), \qquad U_i(D_i - h_i) \le a_i(\delta_j + b_j\delta_k). \tag{1}$$

Put $E_i = L_iU_i > 0$. Multiply these bounds by U_i, L_i and add $E_i\delta_i$:

$$S_i := E_i\delta_i + b_iU_i(\delta_j + a_j\delta_k) + a_iL_i(\delta_j + b_j\delta_k) \ge E_iD_i > 0.$$

Write $S_i = E_i\delta_i + F_i\delta_j + J_i\delta_k$, where

$$F_i = b_iU_i + a_iL_i, \qquad J_i = b_ia_jU_i + a_ib_jL_i.$$

We use the elementary box inequality

$$\prod_i u_i \le 1 + \prod_i v_i + \sum_i u_iv_k \qquad (0 \le u_i \le 1 + v_i, \quad v_i \ge 0).$$

Indeed, the difference is affine in each u_i separately, so its minimum on the box is attained at a vertex. If any $u_i = 0$ it is nonnegative; at the remaining vertex, $u_i = 1 + v_i$ for all i, it is zero. Apply this with $u_i = a_i$, $v_i = a_ib_k$, and then interchange a, b. The slope assumptions give the required bounds on u_i; adding yields

$$H := 2\left(\prod_i a_i - 1\right)\left(\prod_i b_i - 1\right) + \prod_i (a_i + b_i) \ge 0.$$

Set $\eta = (\prod_i a_i)(\prod_i b_i) - 1$, the discrepancy between the product of all six slopes and one. The same square η^2 occurs in each of the following coefficient identities, obtained by direct multiplication:

$$E_iE_j - F_iJ_j = \eta^2, \qquad 2\prod_i E_i - \prod_i F_i - \prod_i J_i = \eta^2 H.$$

Expanding the cubic and collecting these coefficients gives

$$\sum_i E_i\delta_iS_jS_k - \prod_i S_i = \eta^2\left[\sum_i (S_i - E_i\delta_i)\delta_j\delta_k + H\prod_i \delta_i\right] \ge 0. \tag{2}$$

Dividing by $\prod_i S_i > 0$ and using $S_i \ge E_iD_i$ gives $1 \le \sum_i E_i\delta_i/S_i \le \sum_i \delta_i/D_i$. □

5 Completion of the covering theorem

Proof of Theorem 2.1. Suppose a cover has $R < 1$. Normalize the forms. Their linear parts are dependent; reflecting some forms, together with their windows, gives a nonzero relation $\sum_i A_if_i = c$ with $A_i \ge 0$. If necessary, relabel so $A_0 > 0$ and replace f_0 by $g_0 = f_0 - \varepsilon(f_1 + f_2)$ for small $\varepsilon > 0$. Then

$$A_0g_0 + (A_1 + \varepsilon A_0)f_1 + (A_2 + \varepsilon A_0)f_2 = c$$

has all coefficients positive. Widen the first window from $[\ell_0, h_0]$ to $[\ell_0 - 2\varepsilon, h_0]$. Since $0 \le f_1 + f_2 \le 2$ on K, every formerly covered point remains covered. The new directional range is at least $1 - 2\varepsilon$, and

$$R_{\text{new}} \le \frac{R + 2\varepsilon}{1 - 2\varepsilon} < 1$$

for sufficiently small ε. Normalize and clip again. The positive relation persists, with coefficients multiplied by positive directional ranges. This also handles parallel directions.

Keep these forms fixed and select the covering minimum of Section 3. It has strict windows, total relative width below one, and at least six closed exterior cells by Lemma 3.1. Positively rescale the coordinates to put their relation in the form $x_0 + x_1 + x_2 = D$. Lemma 4.2 gives the six mixed cells and six empty same-sign pair caps. Lemma 4.3, including the scalar calculation above, forces $R \geq 1$, a contradiction. □

Thus the covering theorem is complete before any area estimate is used. We turn to the quantitative extension.

6 The quantitative theorem

Write $|A|$ for the area of a measurable set $A \subset \mathbb{R}^2$, and put $V = |K| > 0$. For a family of planks, write G for its uncovered area and $R = \sum_i r_i$ for its total relative width. Put $t_+ = \max\{0, t\}$.

Theorem 6.1 (Three-plank planar QAPC)**.** *If $0 \leq m \leq 3$ and $P_1, \ldots, P_m$ have relative widths $r_1, \ldots, r_m$ with respect to K, then*

$$\boxed{\left| K \setminus \bigcup_{i=1}^{m} P_i \right| \geq |K| \left(1 - \sum_{i=1}^{m} r_i\right)_+^2 .} \tag{3}$$

The coefficient is sharp.

For sharpness, take a triangle with barycentric coordinates $\lambda_1, \lambda_2, \lambda_3$. The support-flush planks $0 \leq \lambda_i \leq r_i$, with $\sum_i r_i \leq 1$, leave a triangle of linear scale $1 - \sum_i r_i$. For fewer planks set the unused widths to zero.

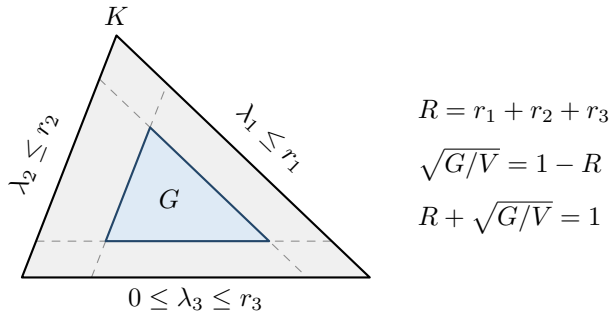


Figure 2: The sharp triangle example motivates the objective. The shaded central triangle has the same shape as K; its linear scale is the unused relative width. The three planks run along the sides of K.

In this example, $\sqrt{G/V} = 1 - R$ is the linear scale of the remaining triangle: it accounts exactly for the unused relative width. This motivates the quantity $C = R + \sqrt{G/V}$. Our aim is to show $C \geq 1$ for every configuration with $R \leq 1$.

The one-plank area inequality alone does not give the sharp bound for several planks. For $R < 1$, it bounds the area covered by plank i by $(2r_i - r_i^2)V$. The union bound would give only

$$G \geq V \left(1 - 2R + \sum_i r_i^2\right),$$

whereas the desired lower bound is $V(1 - 2R + R^2)$. The difference is $2V \sum_{i<j} r_i r_j$, so the proof must control the overlap geometry.

A strict quantitative counterexample gives a minimum of C below one. The covering theorem already proved rules out $G = 0$. At the resulting positive-gap minimum, Brunn–Minkowski and Borsuk–Ulam force at least $2m + 1$ exterior cells. For two planks this is impossible: there are only four sign patterns. For three planks it gives seven of the eight patterns. These supply the pair centres needed for a single window-expansion argument, which gives the quantitative bound. We first establish the area comparisons used in that argument.

6.1 Four exterior corners

Lemma 6.2. *Let K be convex in a real vector space and let f, g be affine forms. If K meets all four weak exterior corner regions of two windows, then every pair of values in those windows is attained at a point of K.*

Proof. Put $F = (f, g)$ and let B be the rectangle of windows. Choose in $F(K)$ one witness from each exterior quadrant of B. Their convex hull contains B: otherwise a strict separator of a missing point would select a quadrant whose witness has at least that point's value, a contradiction. Thus $B \subset F(K)$, which is the required simultaneous attainment. □

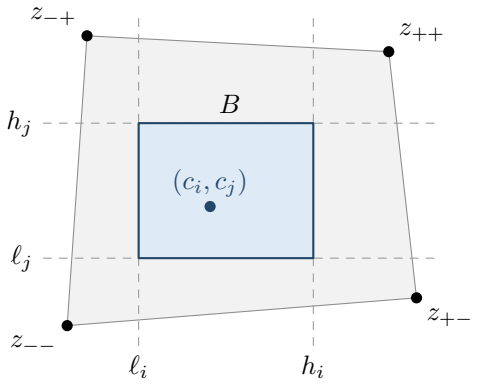


Figure 3: In the image under (f_i, f_j), one witness in each exterior corner forces their convex hull to contain the window rectangle B. In particular, the canonical pair of values (c_i, c_j) is attained in K.

6.2 Window scaling directly from the two caps

Keep a point c fixed in the normalized interval. Contracting a window toward c enlarges its two exterior caps. Brunn–Minkowski controls each cap separately, and adding their areas controls the window.

Lemma 6.3 (Planar window contraction)**.** *Let $K \subset \mathbb{R}^2$ be compact and convex, with area V. Let $f(K) = [0, 1]$, fix $c \in [0, 1]$, and put*

$$A(r) = |K \cap \{(1 - r)c \le f \le c + (1 - c)r\}|.$$

For $t, r \in [0, 1]$,

$$A(tr) \le t^2 A(r) + 2tr(1 - t)V. \tag{4}$$

Proof. The two cap areas at fractional depth s are

$$v_-(s) = |K \cap \{f \le sc\}|, \qquad v_+(s) = |K \cap \{f \ge 1 - s(1 - c)\}|.$$

Here depth one reaches the line $f = c$. For either cap write $v = v_-$ or $v = v_+$, and put $u = \sqrt{v(1)}$, $b = \sqrt{v(1 - r)}$. For the lower cap, a homothety of ratio $1 - r$ about a support point with $f = 0$

takes the depth-one cap into the depth-$(1-r)$ cap; its square-root area is multiplied by $1-r$. Moreover, a convex combination $(1-t)x+ty$ of points in these two caps lies in K and has $f \le (1-tr)c$. Reflection gives the upper-cap versions. Area scaling and Brunn–Minkowski therefore give

$$b \ge (1-r)u, \qquad \sqrt{v(1-tr)} \ge (1-t)u + tb.$$

Squaring the second inequality and applying the first to its cross term yields

$$v(1-tr) \ge t^2 v(1-r) + (1-t^2-2tr(1-t))v(1).$$

Sum over the two caps. Their anchor areas sum to V, and their areas at depth $1-r$ sum to $V - A(r)$, since boundary lines have zero area. Rearranging gives (4). □

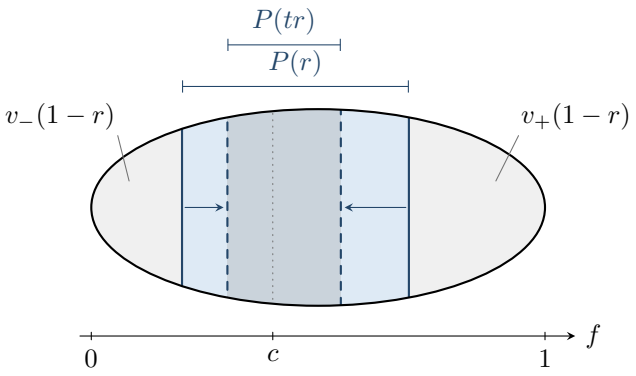


Figure 4: Shrinking the window from width r to tr moves its walls toward $f = c$ and enlarges both exterior caps. At depth one the caps meet at $f = c$, so $v_-(1) + v_+(1) = V$. At depth $1-r$, their total area is $V - A(r)$.

6.3 Expand once to total width one

This estimate does not depend on the number of planks.

Proposition 6.4 (Finite-family estimate)**.** *Let K be a compact convex planar set of area V, and let $P_i = \{\ell_i \le f_i \le h_i\}$, $1 \le i \le m$, where the affine forms have $f_i(K) = [0,1]$, and $0 \le \ell_i \le h_i \le 1$. Put $r_i = h_i - \ell_i$ and suppose $0 < R = \sum_i r_i < 1$. For each pair $i < j$, suppose there is $z_{ij} \in K$ with $f_i(z_{ij}) = c_i$ and $f_j(z_{ij}) = c_j$, where $c_i = \ell_i/(1-r_i)$. If every intersection of three distinct planks has area zero, then the actual uncovered area satisfies*

$$G \ge V(1-R)^2.$$

Since $c_i \in [\ell_i, h_i]$, four exterior corners for each pair supply the required centres by Lemma 6.2.

Proof. Expand the canonical windows to relative widths $\widehat{r}_i = r_i/R$. Write $A_i = |K \cap P_i|$, $B_{ij} = |K \cap P_i \cap P_j|$, and use hats for the expanded quantities. For each pair, the homothety $x \mapsto (1-R)z_{ij} + Rx$ stays in K and satisfies

$$f_i((1-R)z_{ij} + Rx) = (1-R)c_i + Rf_i(x),$$

and likewise for j. It maps each expanded window onto the original window, hence takes the expanded pair intersection into the original one. Area scaling therefore gives $R^2\widehat{B}_{ij} \le B_{ij}$. Apply

window contraction with $t = R$ and sum over i using $\sum_i \widehat{r}_i = 1$. Bonferroni's inequality for the expanded planks inside K gives the second inequality below:

$$\sum_i A_i \le R^2 \sum_i \widehat{A}_i + 2R(1-R)V,$$
$$\sum_i \widehat{A}_i \le V + \sum_{i<j} \widehat{B}_{ij}.$$

The null triple intersections give $G = V - \sum_i A_i + \sum_{i<j} B_{ij}$. Substitution and the pair inequalities yield

$$G \ge V - R^2 V - 2R(1-R)V = (1-R)^2 V.$$

□

7 A minimum with positive uncovered area

For $m = 2, 3$, assume the quantitative bound for $m-1$ planks. The one-plank case is Lemma 6.3 with $r = 1$.

Normalize each affine form f_i to have range $[0,1]$ on K and clip its window to that interval. Clipping preserves gap area and does not increase the relative budget; a disjoint window can be discarded. Put

$$e = (\ell_i, h_i)_{i=1}^m, \qquad r_i = h_i - \ell_i, \qquad R(e) = \sum_i r_i, \qquad V = |K|,$$

and let $G(e)$ denote the actual uncovered area. It is continuous on

$$D = \{e : 0 \le \ell_i \le h_i \le 1,\ R(e) \le 1\}.$$

The target is zero when the original budget is at least one.

On ∂D the bound for $m-1$ planks gives $G \ge V(1-R)^2$. A zero-width plank is null. For a support-flush plank of width $r < 1$, contract K by $1-r$ toward a point on the opposite support face. This copy avoids that plank outside a null boundary line, has area $(1-r)^2 V$, and multiplies all directional widths by $1-r$. Applying the bound for $m-1$ planks to the remaining planks gives

$$(1-r)^2 V \left(1 - \frac{R-r}{1-r}\right)^2 = V(1-R)^2.$$

The faces $R = 1$ and $r = 1$ have zero target.

A strict counterexample therefore makes $C(e) = R(e) + \sqrt{G(e)/V}$ attain a minimum $B < 1$. Its nonempty compact minimum set $Z = \{e \in D : C(e) = B\}$ lies in $\operatorname{int} D$. Choose $p \in Z$ maximizing the Euclidean squared norm of its endpoint coordinates. This choice prevents two symmetric perturbations $p - z$ and $p + z$, with $z \ne 0$, from both being minimizers: their mean squared norm is larger than the squared norm of p. We will use this to count the exterior cells. The covering theorem proved above, together with the known two-plank case, now gives $G(p) > 0$. Indeed, if $G(p) = 0$, the closed planks cover K: any missed point has a neighborhood outside them containing positive body area. Coverage would force $R(p) \ge 1$, contrary to $C(p) < 1$.

The remaining cell count uses the positive gap to normalize one square-root cell-area coordinate. Borsuk–Ulam finds opposite wall motions with proportional area vectors if there are too few cells. Brunn–Minkowski then forces both configurations to minimize C, contradicting the choice of p.

Lemma 7.1. *Let $m = 2$ or $m = 3$, and suppose the quantitative bound holds for $m-1$ planks. A strict m-plank counterexample has an interior minimum p of C with positive uncovered area and at least $2m+1$ nonempty closed exterior cells.*

Proof. Use p, B and Z from the minimum construction above. Retain the k cells nonempty at p, including segments and singletons. Every other cell stays empty near p, by compactness of K. On this neighborhood put

$$b(e) = \left(\sqrt{|C_s(e)|/V}\right)_{s=1}^{k}, \qquad C(e) = R(e) + |b(e)|.$$

Here $|b(e)|$ is the Euclidean norm, equal to $\sqrt{G(e)/V}$. The map b is continuous: moving the endpoints changes cell indicators pointwise outside finitely many null boundary lines, so dominated convergence applies. Choose a small sphere of perturbations z around zero, with both $p \pm z$ in this neighborhood and in D.

We claim that its opposite vectors $b(p - z)$ and $b(p + z)$ cannot be positive scalar multiples. If they were, their zero coordinates would agree. For a positive coordinate, Brunn–Minkowski and $\frac{1}{2}C_s(p - z) + \frac{1}{2}C_s(p + z) \subseteq C_s(p)$ give

$$b_s(p - z) + b_s(p + z) \leq 2b_s(p).$$

This also holds when both left-hand terms vanish. Nonnegativity and proportionality therefore imply

$$|b(p - z)| + |b(p + z)| = |b(p - z) + b(p + z)| \leq 2|b(p)|.$$

Since R is linear, $C(p - z) + C(p + z) \leq 2C(p)$. Minimality forces both endpoints into Z, contradicting the maximal squared norm of p: the squared Euclidean norm is strictly convex, so its average at two distinct endpoints exceeds its value at their midpoint.

Borsuk–Ulam says that every continuous map $S^N \to \mathbb{R}^j$, with $j \leq N$, agrees at some antipodal pair [14, Theorem 2.1.1]. Since $G(p) > 0$, choose a coordinate $b_a(p) > 0$ and shrink the sphere so $b_a(p \pm z) > 0$. If $k \leq 2m$, apply Borsuk–Ulam to

$$z \longmapsto \left(\frac{b_s(p + z)}{b_a(p + z)}\right)_{s \neq a} \in \mathbb{R}^{k-1}.$$

Equal opposite ratios give positive proportional vectors, again contradicting the claim. Hence $k \geq 2m + 1$. □

8 Two planks: five cells would be required

Proposition 8.1. *Two arbitrary planks of total relative width R in a compact convex planar body K leave area at least $|K|(1 - R)_+^2$.*

Proof. A strict counterexample would give a minimum with at least five nonempty exterior cells by Lemma 7.1. Two planks have only four sign patterns, a contradiction. □

9 Three planks: seven cells give the area bound

Suppose a strict three-plank counterexample exists. The two-plank theorem supplies the boundary input for Lemma 7.1. Choose its interior minimum p, with $C(p) < 1$, positive gap, and at least seven of the eight exterior cells present. These supply all four weak exterior corners for every pair: every edge of the cube has a present endpoint (Figure 1(b)). Lemma 6.2 puts each pair's canonical centre inside K.

The strict triple intersection is empty. To see this, take any nonzero affine relation among the three forms and reflect signs so that its coefficients are nonnegative. A point strictly inside all three windows would exclude the all-lower and all-upper closed exterior cells: evaluating the relation at either cell would put its constant strictly below or strictly above its value at that interior point. This contradicts the presence of seven cells. The coefficients need not all be

positive; at least one positive coefficient makes both inequalities strict. Thus no perturbation is needed in the quantitative step.

The closed triple intersection lies in the six boundary lines and has area zero. Proposition 6.4 gives $G \geq V(1-R)^2$, contradicting $C(p) < 1$ and completing Theorem 6.1.

10 The Quantitative Affine Plank Conjecture

The planar theorem suggests the following strengthening of the Affine Plank Conjecture. Here volume is d-dimensional Lebesgue measure.

Conjecture 10.1. *For every $d \geq 1$, every convex body $K \subset \mathbb{R}^d$, and every finite family of planks P_i whose relative widths with respect to K are r_i,*

$$\frac{|K \setminus \bigcup_i P_i|}{|K|} \geq \left(1 - \sum_i r_i\right)_+^d.$$

Support-flush planks on a simplex give equality: in barycentric coordinates $\lambda_0, \ldots, \lambda_d$, remove the regions $\lambda_i \leq r_i$ with $R = \sum_i r_i < 1$. The remaining simplex is obtained by writing $\lambda_i = r_i + (1-R)\eta_i$ and has volume $(1-R)^d|K|$. A single support-flush plank already shows sharpness.

We are not aware of an earlier statement in this generality. The proposed strengthening is the direction-relative, affine-invariant budget R; the exponent and the use of homotheties already appear in earlier work. Write $W = \sum_i \mathrm{width}(P_i)$ for total Euclidean width and $D_{\min}(K)$ for the minimum Euclidean width of K. Bang's erosion and translated-cell argument [8, Lemma 1 and pp. 991–992], in the form of Bakaev–Polyanskii's [4, Corollary 7 and Lemma 8], implies

$$\frac{|K \setminus \bigcup_i P_i|}{|K|} \geq \left(1 - \frac{W}{D_{\min}(K)}\right)_+^d. \tag{5}$$

Indeed, those two results bound the gap volume below by the volume of an erosion containing a translate of $(1 - W/D_{\min}(K))K$ when $W < D_{\min}(K)$. This is a consequence of their machinery, rather than an assertion that Bang stated this formula. Since $R \leq W/D_{\min}(K)$, the affine budget asks for a larger lower bound in general.

K. Bezdek [10, Theorem 5.7] gave an explicit total-gap-volume bound in terms of minimum width and intersections of translates, also extracted from Bang's proof. Zhang–Ding [16, Theorem 1] proved that covering a triangle annulus $T \setminus \varepsilon T$, with the homothetic copy interior to T, requires $W/D_{\min}(T) + \varepsilon \geq 1$. Their statement for triangles constrains the scale of a containing homothetic hole, rather than the volume of an arbitrary gap. These results motivate the conjecture without supplying its direction-relative bound. Our literature search covered the sources cited here and related surveys and problem collections.

Recent stability results address a complementary quantitative question. Bakaev and Yehudayoff [6, Theorems 1.1 and 1.3] prove that a hole of volume v confined to a sufficiently small central homothet εK of an origin-symmetric body in dimension at least two forces excess overlap at least cv/ε, with an absolute $c > 0$; they also resolve Bezdek's concentric-disk-annulus conjecture, giving total width at least two for a unit disk with a sufficiently small concentric disk removed. Ambrus, Corbo, Goto, Leitz and Mawhinney [3, Theorem 1.3] prove total Euclidean width at least one for coverings of unit-cube annuli $Q \setminus \delta Q$, with the homothetic hole interior to Q, when $\delta \leq 1 - 1/\sqrt{d} + \varepsilon_d$ for some $\varepsilon_d > 0$ in dimensions $d = 2, 3$; the proposed range $\delta \leq 1 - 1/d$ remains open. These overlap and annulus bounds concern prescribed locations or shapes of holes. Our estimate bounds an arbitrary gap from below using the relative-width deficit, without symmetry of the planar body.

10.1 Established cases and structural evidence

Proposition 10.2 (Two planks in every dimension)**.** *For every $d \geq 1$, two planks of total relative width R in a convex body $K \subset \mathbb{R}^d$ leave volume at least $|K|(1-R)_+^d$.*

Proof. Write G for the uncovered volume. We first establish the one-plank bound. Normalize its affine form f to range $[0,1]$ on K. After clipping, write the window as $[\ell, \ell + r]$ with $0 \leq \ell \leq \ell + r \leq 1$. The case $r = 1$ is immediate. Otherwise put $c = \ell/(1-r)$ and take support points $a_0, a_1 \in K$ with $f(a_j) = j$. The homotheties $H_j(x) = a_j + (1-r)(x - a_j)$ send K into itself by convexity. Moreover,

$$H_0(K \cap \{f \leq c\}) \subset K \cap \{f \leq \ell\}, \qquad H_1(K \cap \{f \geq c\}) \subset K \cap \{f \geq \ell + r\}.$$

The two source caps partition K up to a null hyperplane. Their images lie outside the window up to null boundary hyperplanes and have disjoint interiors, so $G \geq (1-r)^d |K|$. The simplex example above attains equality.

The minimum argument behind Lemma 7.1 extends to every positive dimension and every finite plank count. The ℓ^d norm of the normalized cell-volume roots is $(G/V)^{1/d}$, so the same objective becomes $C = R + (G/V)^{1/d}$. Assuming the $(m-1)$-plank bound, Brunn–Minkowski, antipodal matching and the same squared-norm tie-break select a counterexample minimum with at least $2m$ nonempty cells, and $2m+1$ if its gap is positive.

For two planks, the one-plank bound supplies the boundary input for this minimum argument. A strict counterexample would therefore give an interior minimum of $C = R + (G/V)^{1/d}$ below one. If $G = 0$, the closed planks cover K, and the two-plank covering theorem [9] forces $R \geq 1$. If $G > 0$, the cell count forces at least five nonempty exterior cells. There are only four sign patterns for two planks. Both alternatives are impossible. □

At most three planks in the plane. This is Theorem 6.1, including arbitrary positions, zero widths and repeated directions.

Constant-width bodies and their affine images. For a body of constant width D, every directional width equals D, so $R = W/D = W/D_{\min}(K)$. Thus (5) proves the conjecture for every finite plank family on such a body. Relative widths and uncovered-volume fractions are preserved by a simultaneous invertible affine change of the body and planks, giving the same conclusion for all its affine images.

A bounded vertex count in the plane. Write q for the uncovered fraction and $\mu_d = q - (1-R)_+^d$ for the defect. A strict counterexample has $\mu_d < 0$, and hence $R < 1$. In dimension two there is a uniform bound on the number of vertices needed to witness such a failure.

Theorem 10.3 (Planar vertex reduction)**.** *For every fixed $m \geq 1$, if a planar m-plank configuration has $\mu_2 < 0$, there is such a configuration on a convex polygon with at most $4m$ vertices and no greater defect. Thus proving the conjectured inequality for these polygons suffices for all planar convex bodies with m planks.*

The replacement may change the planks. The proof in Appendix B combines polygonal replacement, a variation excluding covered edges at a minimum, and a section-endpoint count. That appendix also proves a weaker finite-hull reduction in every dimension: a strict failure persists on a finite hull and on an open set of its parameters. Consequently a full-support sampling law that allows every finite point count assigns positive probability to a failure, if one exists. Neither reduction supplies a finite list of numerical tests or a useful lower bound on that probability.

10.2 What the planar proof does not yet provide

Beyond three planks, the expansion step requires control of intersections of three or more planks: the present pairwise calculation no longer accounts for the union. The covering-minimum step uses three coordinates on a plane and their affine relation to obtain six supporting lines. In higher dimension, or with more planar directions, the cell count does not supply this geometry. These are the two missing steps in extending the present argument.

11 Tight coverings

A covering is *tight* when $R = 1$. Appendix A classifies all tight coverings by two or three planks, including zero widths and repeated directions. The three-plank coverings fall into five families, illustrated in Figure 5: a zero-width member; ordered common chords; an anchored apex; strict triangles; and a split two-direction cover. Theorem A.2 gives the precise forms and proves that the list is exhaustive. The families may overlap.

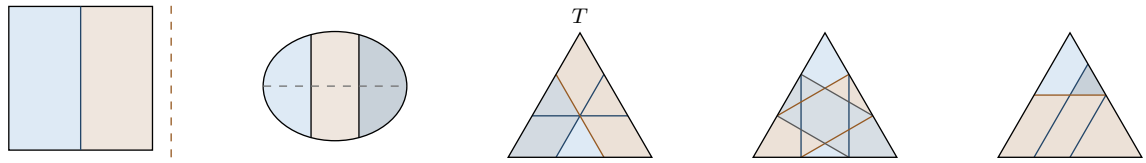


Figure 5: Examples of the five families. I: two consecutive windows cover a square, and the dashed zero-width plank lies outside it. II: a parallel partition illustrates the common-chord family. III: three support-flush planks on a triangle, each of relative width $1/3$, giving the sharp example of Figure 2. IV: the strict-triangle form with $r_0 = r_1 = r_2 = 1/3$. V: split the window $0 \le X \le 1/2$ at $X = 1/4$ in a two-window cover of $X, Y \ge 0$, $X + Y \le 1$. Shading indicates the planks inside the body; families may overlap.

A Tight coverings

A covering is *tight* when its total relative width is one. We give explicit forms for all tight coverings by two or three planks. Throughout, $K \subset \mathbb{R}^2$ is compact and convex with nonempty interior. The proofs are in Sections A.3–A.5 below. This classification concerns covering equality, rather than equality with positive uncovered area in Theorem 6.1.

Write the planks as $\ell_i \le f_i \le h_i$, where the linear part of each affine function f_i is nonzero. Normalize using its actual range:

$$x_i = \frac{f_i - \min_K f_i}{\max_K f_i - \min_K f_i}, \qquad L_i = \frac{\ell_i - \min_K f_i}{\max_K f_i - \min_K f_i}, \quad U_i = \frac{h_i - \min_K f_i}{\max_K f_i - \min_K f_i}.$$

Thus every x_i has attained range $[0, 1]$ on K and the relative width is $r_i = U_i - L_i$. In all the forms below, labels may be permuted and coordinates independently reflected:

$$x_i \longmapsto 1 - x_i, \qquad [L_i, U_i] \longmapsto [1 - U_i, 1 - L_i].$$

These operations retain the exact original endpoints. In particular, a zero-width plank need not meet the body.

A.1 The two-window forms

Two normalized affine coordinates X, Y have actual ranges $[0, 1]$. For positive windows there are the following forms, allowing exchange of X, Y and reflection of either coordinate.

Interior form. For $0 < g, p, q < 1$, the windows are

$$[(1-g)q, (1-g)q+g], \qquad [gp, gp+1-g],$$

and every point of K satisfies

$$\begin{aligned} pq &\le pX + qY, & p(1-q) &\le p(1-X) + (1-q)Y, \\ (1-p)q &\le (1-p)X + q(1-Y), & (1-p)(1-q) &\le (1-p)(1-X) + (1-q)(1-Y). \end{aligned}$$

The four supporting lines enclose the quadrilateral with contacts $(0,p), (q,0), (1,p), (q,1)$.

Anchored form. For $0 < g < 1$ and $0 \le c \le 1$, the windows are

$$[0, g], \qquad [gc, gc+1-g],$$

and the body satisfies

$$cX \le Y \le 1 - (1-c)X. \tag{6}$$

This is an enclosure together with the stipulated actual ranges; it is not an extra covering assumption. The endpoint cases $c = 0, 1$ are included. For the complete two-window list also allow a zero-width window at any position and the full window $[0, 1]$ in the other coordinate.

Lemma A.1. *These are exactly the tight two-window coverings of a convex set with attained unit ranges.*

A.2 The five three-window families

The following list uses only explicit parameters and body conditions. Neither a covering predicate nor an assumed total-width equality is hidden in its definitions.

I. A zero-width member. After relabeling, $L_0 = U_0$ is arbitrary and the remaining two windows belong to the complete list of Lemma A.1. This includes two arbitrarily positioned zero-width windows and one full window.

II. Ordered common chords. Choose $0 < r < s < 1$. The windows are

$$[0, r], \quad [r, s], \quad [s, 1],$$

and $x_0 \le x_1 \le x_2$ throughout the body. The unit ranges imply points where all three coordinates are zero and where all three are one. The segment joining these points is a common extremal chord. Parallel consecutive partitions are included, by taking $x_0 = x_1 = x_2$.

III. An anchored apex. The coordinate image of the body is the triangle $\mathrm{conv}\{T, A, B\}$, where $T_0 = 1$ and $A_0 = B_0 = 0$. Choose $0 < r < 1$ and a positive two-window form $[l, u], [m, v]$ for coordinates x_1, x_2 on this same body. The three windows are

$$[0, r], \quad [rT_1 + (1-r)l,\ rT_1 + (1-r)u], \quad [rT_2 + (1-r)m,\ rT_2 + (1-r)v]. \tag{7}$$

The two-window bounds include $0 \le l < u \le 1$ and $0 \le m < v \le 1$.

IV. Strict triangles. The body is the convex hull of three affinely independent vertices v_0, v_1, v_2. Choose $r_i > 0$ with $r_0 + r_1 + r_2 = 1$. With cyclic indices,

$$L_i = r_{i+1}, \qquad U_i = 1 - r_{i+2}, \qquad x_i(v_i) = \frac{r_{i+1}}{1-r_i}, \quad x_i(v_{i+1}) = 0, \quad x_i(v_{i+2}) = 1. \tag{8}$$

These equations specify the actual coordinates and endpoints, with two free width parameters. Every endpoint is strictly inside its range.

V. A split two-direction cover. Two oriented normalized functions agree, say $x_0 = x_1$, and the third direction is independent. All three windows have positive widths. Either $U_0 = L_1$ and the pair $[L_0, U_1], [L_2, U_2]$ is a positive two-window form, or $U_1 = L_0$ and that pair is $[L_1, U_0], [L_2, U_2]$. Thus one window of a tight two-window covering is subdivided at a strictly interior point.

Theorem A.2 (Complete equality classification)**.** *Three ordered closed plank windows on a compact planar convex body with nonempty interior form a tight covering if and only if, after the stated permutations and reflections, they belong to one of families I–V. The families need not be disjoint.*

A.3 Two supporting calculations

We use the normalization and families of Appendix A. The two preparatory calculations below retain the supporting arguments needed for equality; the quantitative minimum proof alone does not classify arbitrary tight covers.

Lemma A.3 (Two planks)**.** *Two planks covering a convex set with attained positive directional ranges have total relative width at least one. The same statement holds for two affine windows on a convex subset of any real vector space.*

Proof. Normalize both directional ranges to $[0, 1]$, clip both windows to that interval, and translate them to $[0, r_0]$ and $[0, r_1]$. If either clipped width is one, the conclusion is immediate. Otherwise $r_0, r_1 < 1$. The four coordinate contacts of the convex image K are

$$L = (-a, s_0), \quad R = (r_0 + b, t_0), \quad B = (p_0, -c), \quad T = (q_0, r_1 + d),$$

where $a, b, c, d \geq 0$, $a + b = 1 - r_0$ and $c + d = 1 - r_1$. Clamp s_0, t_0 to $[0, r_1]$ and p_0, q_0 to $[0, r_0]$, obtaining s, t, p, q. The four original contact segments give

$$ac \leq ps, \qquad bc \leq (r_0 - p)t, \qquad ad \leq q(r_1 - s), \qquad bd \leq (r_0 - q)(r_1 - t). \tag{9}$$

For example, if $ac > 0$, coverage puts $s_0 \in [0, r_1]$ and $p_0 \in [0, r_0]$, so neither is clamped. Along LB, entering the x-window must precede leaving the y-window: $a/(a + p) \leq s/(s + c)$, hence $ac \leq ps$. If $ac = 0$, the inequality is automatic. Reflection gives the other three. Cauchy–Schwarz gives

$$c(a + b)^2 \leq r_0(as + bt), \qquad d(a + b)^2 \leq r_0\big(a(r_1 - s) + b(r_1 - t)\big). \tag{10}$$

For example, the first follows from

$$\sqrt{c}\,(a + b) \leq \sqrt{p}\,\sqrt{as} + \sqrt{r_0 - p}\,\sqrt{bt} \leq \sqrt{r_0(as + bt)}.$$

Add (10) and cancel $a + b > 0$ to obtain $(1 - r_0)(1 - r_1) \leq r_0 r_1$, or $r_0 + r_1 \geq 1$. The displayed inequalities also hold when $r_0 + r_1 = 1$; their equality conditions will be used below. □

Lemma A.4 (Choosing supports for a finite contact hull)**.** *Let F be a finite subset of $x_0 + x_1 + x_2 = D$ whose convex hull has actual coordinate ranges $[0, D_i]$, and suppose $0 < \ell_i < h_i < D_i$. If all six strict same-sign pair caps are empty, one can choose positive supporting slopes a_i, b_i satisfying*

$$a_{i+1}(1 - b_i) \leq 1, \qquad b_{i+1}(1 - a_i) \leq 1. \tag{11}$$

Proof. Separation as in Lemma 4.1 supplies positive slopes for the rows

$$x_i + a_i x_{i+1} \leq h_i + a_i h_{i+1}, \qquad x_i + b_i x_{i+1} \geq \ell_i + b_i \ell_{i+1}.$$

For each row, the slopes feasible on F form a closed interval. Each individual strict cut meets F, by the actual ranges and strict endpoints. A point in one cut bounds the slope away from zero;

a point in the other bounds it above. Thus this interval is nonempty, compact and contained in $(0, \infty)$. Choose its point nearest to one.

Put $j = i + 1$ and $k = i + 2$, cyclically. The first inequality in (11) is immediate if $a_j \le 1$ or $b_i \ge 1$. Otherwise $a_j > 1$ and $b_i < 1$. Moving a_j down toward one is blocked by a point $P \in F$ where its row is tight and $P_j > h_j$. Moving b_i up is blocked by a point $Q \in F$ where its row is tight and $Q_j < \ell_j$. Indeed, a finite family of linear inequalities can stop such a motion only at a binding inequality with this sign. Set $\tau = P_j - Q_j > 0$, $u = P_i - Q_i$ and $v = Q_k - P_k$. Evaluate the upper row at Q and the lower row at P, subtracting their equalities at P and Q. This gives

$$a_j v \le \tau, \qquad -b_i \tau \le u.$$

The plane gives $u + \tau = v$. Hence $a_j(1 - b_i)\tau \le a_j(u + \tau) = a_j v \le \tau$, and cancellation proves $a_j(1 - b_i) \le 1$. Reflecting all coordinates interchanges the upper and lower rows and proves the second inequality. □

A.4 Two-window rigidity and repeated directions

Proof of Lemma A.1. Clipping a tight pair cannot shorten either window, by Lemma A.3. For two positive interior windows, equality holds throughout the contact-segment and Cauchy–Schwarz inequalities (9)–(10). Put $g = u - l$, $q = l/(1 - g)$ and $p = m/g$, where $[l, u], [m, v]$ are the windows. The equalities force the four contacts to be $(0, p), (q, 0), (1, p), (q, 1)$. Each uncovered corner is empty. Separating that corner from the body, and using its two adjacent contacts, gives the displayed supporting line. This gives all four inequalities of the interior form.

If $l = 0$, the cap $X > g$ is covered by the other window of width $1 - g$. Segments from a point with $X = 1$ to points with $Y = 0, 1$ force that window to be $[gc, gc + 1 - g]$ for some $c \in [0, 1]$. Applying the same segment argument to any body point gives (6). Reflection and exchange treat the other three boundary choices. A zero-width window can be deleted by closedness, leaving a full window.

Conversely, in the anchored form, $X > g$ implies $gc \le Y \le gc + 1 - g$. In the interior form, each of the four strict outside corners contradicts its corresponding displayed inequality. Thus the windows cover. Their widths add to one in every listed form. □

Lemma A.5 (A repeated direction)**.** *If exactly two plank directions agree and all three widths are positive, a tight covering belongs to family V.*

Proof. Orient the repeated coordinate as X and the other as Y. Write the windows as $[l_0, u_0], [l_1, u_1]$ in X and $[a, b]$ in Y. Clipping and Theorem 2.1 place all endpoints in $[0, 1]$. Order $l_0 \le l_1$. Containment or interior overlap would let us merge the two X windows and obtain a two-window cover of total width less than one. Thus $u_0 \le l_1$.

Suppose $u_0 < l_1$. Each of the convex caps $Y < a$ and $Y > b$ lies entirely in one X window, since a convex set cannot cross the gap between them. If both caps lie in the same window, that window and $[a, b]$ already cover K, again with total width less than one. After reflecting Y if necessary, the lower cap lies in $[l_0, u_0]$ and the upper cap in $[l_1, u_1]$.

The two-window bound applied to the lower cap together with $Y \ge a$ gives $u_0 - l_0 \ge a$. Its reflected version gives $u_1 - l_1 \ge 1 - b$. Tightness forces equality in both. Since the two X widths are positive, $0 < a < b < 1$. The anchored form of Lemma A.1, applied to these two auxiliary covers, supplies $c, d \in [0, 1]$ such that

$$\begin{aligned} l_0 &= (1 - a)c, & u_0 &= (1 - a)c + a, & c(1 - Y) \le X \le c + (1 - c)Y, \\ l_1 &= bd, & u_1 &= bd + 1 - b, & dY \le X \le 1 - (1 - d)Y. \end{aligned}$$

The gap gives $bd > u_0 > 0$, hence $d > 0$. At an $X = 0$ contact, the second lower bound forces $Y = 0$, and the first then forces $c = 0$. At an $X = 1$ contact, the first upper bound now forces

$Y = 1$, and the second forces $d = 1$. The two enclosures reduce to $Y \le X \le Y$ throughout K, impossible for independent coordinates on a body with nonempty interior. Thus the two windows touch. Merging them and applying Lemma A.1 proves the assertion. □

A.5 Why every case is included

We first isolate the two geometric rigidity statements used for pairwise distinct directions. They concern the actual body, rather than a chosen six-point contact hull. Reflecting exact range coordinates puts their image on a plane $a_0x_0 + a_1x_1 + a_2x_2 = D$ with every $a_i > 0$.

Lemma A.6 (Boundary rigidity)**.** *A tight cover with three pairwise distinct directions, three positive widths, and at least one endpoint at a support level belongs to family II or III.*

Proof. Tightness and clipping place every positive window within its actual range. After relabeling and, if necessary, reflecting all three coordinates, write the anchored window as $0 \le X \le r$, with $0 < r < 1$. Write the other coordinates as Y, Z, so that $aX + bY + cZ = D$, where $a, b, c > 0$. For any $T \in K$ with $X(T) = 1$, contract by

$$H_T(z) = rT + (1 - r)z.$$

If $X(z) > 0$, then $X(H_T(z)) > r$, so the other two planks cover $H_T(z)$. Density and closedness extend this to every $z \in K$. Pulling their windows back gives a two-window cover of total width one. Apply Lemma A.1.

We shall repeatedly use the following elementary corner observation. Suppose C lies strictly between $A, B \in K$, at the intersection of two endpoint lines of this pulled-back cover. If $X(C) > 0$, then T lies on the line AB. To see this, let g, h be the two affine outside cuts, zero at C. Coverage excludes

$$X > r, \qquad g > rg(T), \qquad h > rh(T).$$

Separation from the open positive octant gives a nonzero nonnegative combination of these three cuts which is nonpositive on K. At $H_T(C)$ the first cut is strictly positive and the other two are zero, so the first coefficient vanishes. Evaluation at T and C then gives

$$\alpha g(T) + \beta h(T) = 0, \qquad \alpha g + \beta h \le 0 \text{ on } K, \quad \alpha, \beta \ge 0, \quad \alpha + \beta > 0.$$

The supporting line contains C and hence A, B, as C lies strictly between them. It also contains T, proving the observation.

The interior two-window form is impossible. Its attained ranges force the four contacts

$$L = (0, p), \quad R = (1, p), \quad B = (q, 0), \quad U = (q, 1)$$

in the (Y, Z) plane. Comparing B with R and L with U in the positive plane relation gives $X(B) > 0$ and $X(L) > 0$. Apply the corner observation to $gR + (1 - g)B$ and $gL + (1 - g)U$. It gives

$$\begin{aligned} p(1 - Y(T)) + (1 - q)Z(T) &= p(1 - q), \\ (1 - p)Y(T) + q(1 - Z(T)) &= (1 - p)q. \end{aligned}$$

Consequently $Y(T) \ge q > 0$ and $Z(T) \ge p$. But $X(T) = 1 \ge X(L)$, so the plane expression at T is strictly larger than at L, a contradiction.

It also follows that T is unique. For every point in the face $X = 1$, the pulled-back pair has an anchored endpoint. Thus either $rY(T)$ is one of the two fixed values $L_1, U_1 + r - 1$, or $rZ(T)$ is one of $L_2, U_2 + r - 1$. Each such value specifies at most one point of the line $a + bY + cZ = D$. The face is therefore finite and convex, hence a singleton.

An interior triangular enclosure gives an apex. Allow reflection and exchange of Y, Z within this paragraph. When the anchored two-window parameter satisfies $0 < c_0 < 1$, the attained ranges force the actual body to be the triangle

$$A = (0, 0), \quad B = (0, 1), \quad C = (1, c_0).$$

The unique maximizer T of X is one of its vertices. If $T = A$ or B, the corner observation on the opposite edge forces $X = 0$ at both its ends: otherwise its interior corner would have positive X and would force T onto that edge line. If $T = C$, consider $H_T(A)$. Its Y coordinate is the lower endpoint of the first pulled-forward window, while its Z coordinate is strictly below the second window. If $X(A) > 0$, moving slightly toward A misses all three windows. Thus $X(A) = 0$. The same argument at $H_T(B)$, above the second window, gives $X(B) = 0$. This is family III, with the exact endpoints (7).

The endpoint triangular enclosures. In the original Y, Z orientations, these are

$$Y + Z \le 1, \qquad Y + Z \ge 1, \qquad Y \le Z, \qquad Z \le Y.$$

For $Y \le Z$, attained ranges supply $(Y, Z) = (0, 0)$ and $(1, 1)$. The positive plane relation makes these respectively the maximum and minimum of X, so their X values are 1 and 0. They are opposite vertices of the unit cube. The case $Z \le Y$ is identical.

Suppose $Y + Z \le 1$. Let $R = (1, 0)$ and $U = (0, 1)$ be the forced contacts. A point with $X = 0$ gives $D \le \max(b, c)$; the two contacts give $D \ge \max(b, c)$. Thus at least one of $X(R), X(U)$ is zero. The pulled-back windows are $[0, g]$ and $[0, 1 - g]$. If both contact values are zero, then $b = c = D$. Separation at $H_T(U)$, where the X and Y endpoint cuts are equalities and the Z window is strictly missed, gives $Y \ge Y(T)X$ on K. Indeed, the separating coefficient of the strict cut is zero, and the remaining support line contains both T and U. Similarly $Z \ge Z(T)X$. Every point of K now has the nonnegative barycentric coordinates

$$X, \qquad Y - Y(T)X, \qquad Z - Z(T)X$$

relative to T, R, U; the plane equation makes their sum one. So $K = \mathrm{conv}\{T, R, U\}$ with a zero X base, giving family III. If the two contact values are not both zero, the corner observation on RU gives $Y(T) + Z(T) = 1$. Thus $T \in RU$; since $X(T) = 1$ and one endpoint has $X = 0$, the other endpoint has $X = 1$. Again R, U are opposite cube vertices.

Finally suppose $Y + Z \ge 1$. The contacts are now $R = (0, 1)$ and $U = (1, 0)$. The positive plane relation and $X(T) = 1$ show that at least one has $X = 1$, so uniqueness makes it T. Assume $T = R$, exchanging Y, Z if needed. The two actual windows are

$$[(1 - r)(1 - g), 1 - r], \qquad [r + (1 - r)g, 1].$$

If $X(U) > 0$, the point $H_T(U)$ lies beyond the first plank and strictly below the Z window, with $Y = 1 - r$. Moving slightly toward U also misses the Y window, a contradiction. Hence $X(U) = 0$, again giving opposite cube vertices.

In every opposite-vertex case, reflect coordinates to put the common extremal chord between $(0, 0, 0)$ and $(1, 1, 1)$. Restricting the windows to this chord partitions $[0, 1]$ into $[0, r], [r, s], [s, 1]$ after relabeling. To obtain the body inequalities, separate K from the empty corner $X > r, Y < r, Z < s$. At the chord point (r, r, r) the last cut is strict, so its coefficient vanishes. Evaluation at the two chord endpoints makes the other coefficients equal and positive, giving $X \le Y$ throughout K. Reflecting this argument at (s, s, s) gives $Y \le Z$. This is family II, completing all boundary cases. □

Lemma A.7 (A common point at equality)**.** *In a tight cover with three positive strictly interior windows and three distinct directions, every pair meets in the body outside the third plank. All three planks have a common point in the body. Consequently, in the positive plane coordinates,*

$$\sum_i a_i L_i \le D \le \sum_i a_i U_i.$$

Proof. Suppose the intersection of two planks inside K lies in the third. Write the third window as $a \le Y \le b$. Each of the connected caps $Y < a$ and $Y > b$ is covered by the other two closed planks, which are disjoint on that cap. Each cap therefore lies entirely in one of them. If both caps lie in the same plank, two windows cover K, contradicting their total width being less than one.

Otherwise, after relabeling and reflection, the lower cap lies in an X window and the upper cap in a Z window. The two-window bound, applied to the auxiliary covers using $Y \ge a$ and $Y \le b$, gives $r_X \ge a$ and $r_Z \ge 1 - b$. Tightness forces both equalities. The anchored two-window form then gives $c, d \in [0, 1]$ with

$$c(1 - Y) \le X \le c + (1 - c)Y, \qquad dY \le Z \le 1 - (1 - d)Y.$$

The strict endpoints of the X window make $0 < c < 1$. Thus both an $X = 0$ contact and an $X = 1$ contact have $Y = 1$, and the second enclosure makes their Z coordinates equal to d. This contradicts the nonzero coefficient of X in the plane equation. It proves the first assertion, and in particular pairwise intersection.

For completeness, three closed convex sets covering a compact convex set and meeting pairwise inside it have a common point. Clip the three sets to K and call them A, B, C. If their triple intersection were empty, strictly separate the compact convex sets $A \cap B$ and C by an affine function h, negative on the former and positive on the latter. Choose $r \in A \cap B$, $p \in A \cap C$, $q \in B \cap C$. The segments rp, rq meet $h = 0$ at points of A, B, respectively. The segment joining these two points lies in K and misses C. It is therefore covered by the closed sets A, B; connectedness gives a point in both, contradicting $h < 0$ on $A \cap B$. Apply this observation to the three planks. Their common point, evaluated in the positive plane equation, gives the two displayed bounds. □

Lemma A.8 (Interior rigidity)**.** *A tight cover with three pairwise distinct directions and* $0 < L_i < U_i < 1$ *for every* i *has an actual triangular body.*

Here is the finite reduction for this lemma. In circular order, set

$$(g_0, \dots, g_5) = (x_0 - U_0, L_1 - x_1, x_2 - U_2, L_0 - x_0, x_1 - U_1, L_2 - x_2).$$

Subscripts on these six cuts are taken modulo six. Each triple $g_{i-1} > 0, g_i > 0, g_{i+1} > 0$ is an uncovered mixed sign pattern. The planar separation argument of Lemma 4.1 makes at least one of its three pair caps empty. Record the empty cap $g_{i-1} > 0, g_{i+1} > 0$ as a singleton at node i, and the empty cap $g_i > 0, g_{i+1} > 0$ as edge i. Thus singletons and edges cover the six-cycle.

An inclusion-minimal such cover consists of selected edges and a singleton at every vertex not incident with them. Three consecutive selected edges are impossible, since the middle edge would be redundant. Conversely, every edge selection with no three consecutive edges, together with those singletons, is minimal. Up to rotation and reflection, the possibilities are exactly the following; S lists singleton nodes and E lists edges.

Case	S	E	Number of labeled patterns
A	$\{0, 1, 2, 3, 4, 5\}$	$\varnothing$	1
B	$\{2, 3, 4, 5\}$	$\{0\}$	6
C	$\{3, 4, 5\}$	$\{0, 1\}$	6
D	$\{4, 5\}$	$\{0, 2\}$	6
E	$\{2, 5\}$	$\{0, 3\}$	3
F	$\{5\}$	$\{0, 1, 3\}$	12
G	$\varnothing$	$\{0, 2, 4\}$	2
H	$\varnothing$	$\{0, 1, 3, 4\}$	3

This enumeration follows by the number of selected edges: zero or one; the three separations for two edges; a two-edge path and an isolated edge, or three alternating edges; and two two-edge paths. Five selected edges would contain three consecutive edges. Thus there are 39 labeled patterns, with no geometric cases omitted by the enumeration. Figure 6 displays the eight representatives.

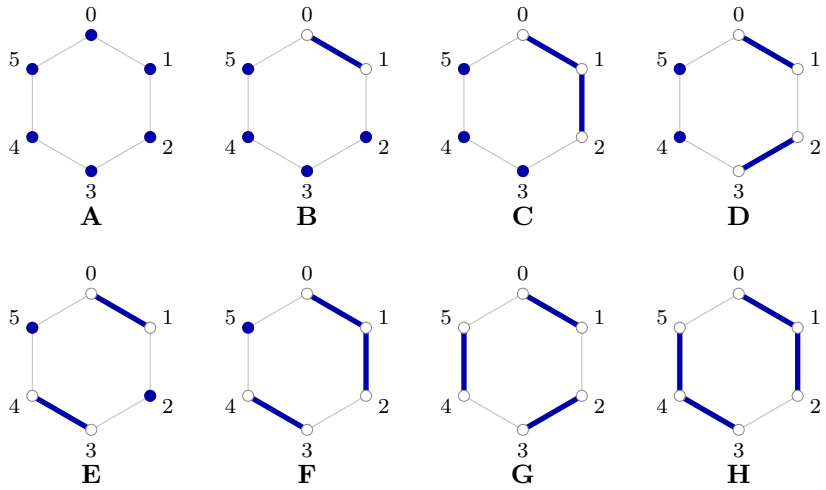


Figure 6: The eight minimal cap patterns A–H on the six-cycle. A filled vertex selects a singleton in S; a thick edge selects an edge in E. Every vertex is selected or incident to a selected edge. The hexagons depict the cap patterns, not the shape of the body.

We give the following exclusions directly. Write $r_i = U_i - L_i > 0$, so that $\sum_i r_i = 1$.

Case E. All four outside corners for coordinates x_0, x_1 are empty. Hence their two windows already cover K, contradicting the two-window bound because $r_0 + r_1 < 1$.

Case H. Put $Y = x_1$ and let $J = \{0, 2\}$. The four mixed caps give

$$Y > U_1 \implies x_j \geq L_j \ (j \in J), \qquad Y < L_1 \implies x_j \leq U_j \ (j \in J).$$

Choose $i \in J$ with $a_i = \max_{j \in J} a_j$. On the upper cap, the plane equation and the lower bounds for both x_j put x_i in an interval of length at most

$$\frac{D - a_1 U_1 - \sum_{j \in J} a_j L_j}{a_i}.$$

This length is at least $1 - U_1$: contract K toward a $Y = 1$ contact by a factor $1 - s$ with $s > U_1$, and let $s \downarrow U_1$. The contracted body lies in the upper cap and has x_i-range $1 - s$. The reflected argument in the lower cap gives

$$a_i(1 - U_1) \leq D - a_1 U_1 - \sum_{j \in J} a_j L_j,$$
$$a_i L_1 \leq a_1 L_1 + \sum_{j \in J} a_j U_j - D.$$

Adding these inequalities yields

$$a_i(1 - r_1) \leq \sum_{j \in J} a_j r_j - a_1 r_1 \leq a_i(1 - r_1) - a_1 r_1,$$

impossible since $a_1 r_1 > 0$.

Case D. Separation of its four empty caps gives positive a, b, c, d such that throughout K, writing $(X, Y, Z) = (x_0, x_1, x_2)$,

$$\begin{aligned} X + aY &\le U_0 + aU_1, & L_2 + bL_0 &\le Z + bX, \\ X - cY &\le U_0 - cL_1, & Z - dX &\le U_2 - dL_0. \end{aligned}$$

Put $A = a + c$ and $B = b + d$. Evaluate these supports at the six attained coordinate extrema, eliminating the other coordinates. In the order $X = 0, 1$, $Y = 0, 1$, $Z = 0, 1$, the resulting inequalities are

$$\begin{aligned} BL_0 &\le r_2, & A(1 - U_0) &\le acr_1, \\ cBL_1 &\le Br_0 + r_2, & aB(1 - U_1) &\le Br_0 + r_2, \\ AL_2 &\le bAr_0 + abcr_1, & A(1 - U_2) &\le dAr_0 + acdr_1. \end{aligned}$$

For example, the two bounds on Z give $B(L_0 - X) \le r_2$; at $Y = 0$ combine this with $X \le U_0 - cL_1$ to get the third row. The other rows follow by the same additions of the displayed supports. Set $N = ABr_0 + acBr_1 + Ar_2$. Combining each pair of rows gives

$$AB \le N, \qquad acB \le N, \qquad A \le N.$$

Since N is a convex combination of these three numbers with strictly positive weights r_i, each equals N. Every row used in the first two combinations is therefore an equality. In particular,

$$BL_0 = r_2, \qquad cBL_1 = Br_0 + r_2, \qquad aB(1 - U_1) = Br_0 + r_2.$$

Thus $cL_1 = U_0 = a(1 - U_1)$. At each of the contacts $Y = 0$ and $Y = 1$, the upper supports now force $X = 0$. The lower and upper Z supports coincide there, since $BL_0 = r_2$. The two contacts have the same X, Z coordinates and distinct Y coordinates, contradicting the original plane equation whose coefficient a_1 is positive.

Case C. The five empty caps give positive α, β, b, c, d with

$$\begin{aligned} X + \alpha Y &\le U_0 + \alpha U_1, & Y + \beta Z &\le U_1 + \beta U_2, \\ L_2 + bL_0 &\le Z + bX, & X - cY &\le U_0 - cL_1, \\ Z - dY &\le U_2 - dL_1. \end{aligned}$$

Eliminating coordinates at the six attained extrema, in the same order as in case D, gives

$$\begin{aligned} b(1 + \beta d)L_0 &\le (1 + \beta d)r_2 + dr_1, & (\alpha + c)(1 - U_0) &\le \alpha cr_1, \\ (d + bc)L_1 &\le r_2 + br_0, & (1 + \alpha\beta b)(1 - U_1) &\le \beta r_2 + \beta br_0, \\ (\alpha + c)L_2 &\le b(\alpha + c)r_0 + \alpha bcr_1, & (1 + \beta d)(1 - U_2) &\le dr_1. \end{aligned}$$

For example, the two mixed supports and the lower support give $(d + bc)(L_1 - Y) \le r_2 + br_0$, which gives the third inequality at $Y = 0$. Set

$$Q = \frac{b\alpha c}{\alpha + c} + \frac{d}{1 + \beta d}, \qquad V = \frac{1}{d + bc} + \frac{\beta}{1 + \alpha\beta b}.$$

The identity

$$(1 - QV)(\alpha + c)(1 + \beta d)(d + bc)(1 + \alpha\beta b) = b(\alpha\beta d - c)^2$$

shows that $QV \le 1$. Pairing the contact inequalities gives

$$\begin{aligned} 1 &\le r_0 + (Q/b)r_1 + (1/b)r_2, \\ 1 &\le r_1 + Vr_2 + bVr_0, \\ 1 &\le r_2 + br_0 + Qr_1. \end{aligned}$$

For $T = br_0 + Qr_1 + r_2$, the first and third inequalities give $b \le T$ and $1 \le T$. Multiplying the second by Q and using $QV \le 1$ gives $Q \le T$. Again T is a convex combination with all weights positive, so $b = Q = 1$. The middle inequality then forces $V = 1$, and the square identity gives $c = \alpha\beta d$.

The two middle-coordinate contact bounds now sum to equality, since $V = 1$ and $L_1 + (1 - U_1) = r_0 + r_2$. Thus each is saturated:

$$(d + c)L_1 = r_0 + r_2, \qquad 1 - U_1 = \beta d L_1.$$

At $Y = 0$, the mixed supports give $X \le x_* := U_0 - cL_1$ and $Z \le z_* := U_2 - dL_1$. Their sum equals $L_0 + L_2$, so the lower support forces both equalities. At $Y = 1$, the two upper supports give the same bounds, because $\alpha(1 - U_1) = cL_1$ and $(1 - U_1)/\beta = dL_1$. The lower support again forces equality. The $Y = 0$ and $Y = 1$ contacts therefore have the same X, Z, contradicting the positive plane relation.

Case B. Write the five positive-slope supports as

$$\begin{aligned} X + aY &\le U_0 + aU_1, & Y + bZ &\le U_1 + bU_2, \\ L_0 + pL_1 &\le X + pY, & L_2 + qL_0 &\le Z + qX, \\ X - cY &\le U_0 - cL_1. \end{aligned}$$

All five slopes a, b, p, q, c are positive. Put $D_p = 1 + pbq$ and $D_a = 1 + abq$. The six contact inequalities are

$$\begin{aligned} D_pL_0 &\le pr_1 + pbr_2, & (a + c)(1 - U_0) &\le acr_1, \\ (p + c)L_1 &\le r_0, & D_a(1 - U_1) &\le br_2 + bqr_0, \\ (a + c)L_2 &\le q(a + c)r_0 + acqr_1, & b(p + c)(1 - U_2) &\le (p + c)r_1 + r_0. \end{aligned}$$

As before, these follow by eliminating the other two coordinates at each attained extremum. Pairing them gives

$$1 \le d_i := r_i + f_i r_{i+1} + g_i r_{i+2},$$

where

i	f_i	g_i
0	$p/D_p + ac/(a + c)$	pb/D_p
1	b/D_a	$1/(p + c) + bq/D_a$
2	$q + 1/(b(p + c))$	$qac/(a + c) + 1/b$

Every coefficient is positive. Direct subtraction gives

$$\begin{aligned} 1 - f_0g_1 &= \frac{(c - apbq)^2}{(a + c)(p + c)D_aD_p}, \\ 1 - f_1g_2 &= \frac{a^2bq}{(a + c)D_a}, & 1 - f_2g_0 &= \frac{c}{(p + c)D_p}, \\ g_1 - f_1f_2 &= \frac{abq}{(p + c)D_a}, & f_0 - g_0g_2 &= \frac{ac}{(a + c)D_p}. \end{aligned}$$

These inequalities have a short weighted-average consequence. Set

$$v_0 = 1, \qquad v_1 = f_0, \qquad v_2 = \max(g_0, f_0f_1), \qquad T = \sum_i r_i v_i. \tag{12}$$

They imply $f_iv_i \le v_{i+1}$ and $g_iv_i \le v_{i+2}$ for every i. For example, $g_2v_2 \le f_0$ follows from $g_0g_2 \le f_0$ and $f_1g_2 \le 1$. Moreover $f_2v_2 < 1$, since both $f_2g_0 < 1$ and $f_0f_1f_2 < f_0g_1 \le 1$. Thus

$$v_i \le v_i d_i \le T \quad (i = 0, 1, 2), \qquad v_2 < T.$$

But T is a convex combination of the v_i with all weights positive. It cannot be at least every v_i and strictly greater than one of them. This excludes case B.

Case F. Write the plane equation in this paragraph as $\sum_i A_i x_i = D$, where $A_i > 0$. The four supports have positive slopes a, c, d, e:

$$X + aY \le U_0 + aU_1, \qquad X - cY \le U_0 - cL_1,$$
$$Z - dY \le U_2 - dL_1, \qquad Y - eX \le U_1 - eL_0.$$

By Lemma A.7, $P := D - \sum_i A_i L_i \ge 0$. Set

$$M = A_1 + A_2 d, \quad \kappa = A_0 + eM, \quad \Lambda = A_0 c + M, \quad t = \frac{ac}{a+c}, \quad s = \frac{e}{1+ae}.$$

Define two positive coefficients by

$$(C_0, C_1) = \begin{cases} (A_0,\ a(A_0c + A_1)/(a+c)), & A_1 \le aA_0, \\ ((A_0 + eA_1)/(1+ae),\ A_1), & A_1 \ge aA_0. \end{cases}$$

The choices agree when equality holds. In either case,

$$\begin{aligned} C_0 &\le \min(A_0 + eA_1,\ A_0 + s\Lambda), \\ C_1 &\le \min(A_0 c + A_1,\ tA_0 + A_1). \end{aligned} \tag{13}$$

These follow immediately by dividing by $a + c > a$ or $1 + ae > 1$. The contact inequalities are

$$\kappa L_0 \le Mr_1 - P + A_2 r_2, \qquad (a+c)(1-U_0) \le acr_1,$$
$$\Lambda L_1 \le A_0 r_0 - P + A_2 r_2, \qquad (1+ae)(1-U_1) \le er_0,$$
$$A_2 L_2 \le C_0 r_0 + C_1 r_1 - P, \qquad \kappa(1-U_2) \le ed(P - A_2 r_2) + A_0 d r_1.$$

They follow from the supports and the plane equation at the six extrema. For the $Z = 0$ inequality, if $A_1 \le aA_0$, combine the first two supports with weights $A_1 + A_0 c$ and $aA_0 - A_1$; if $A_1 \ge aA_0$, combine the first and fourth with weights $A_0 + eA_1$ and $A_1 - aA_0$. This explains both choices of C_0, C_1.

Put $d_i = r_i + f_i r_{i+1} + g_i r_{i+2}$, where

i	f_i	g_i
0	$t + M/\kappa$	A_2/κ
1	A_2/Λ	$s + A_0/\Lambda$
2	C_0/A_2	$C_1/A_2 + A_0 d/\kappa$

Pairing the first four contact inequalities gives $d_0, d_1 \ge 1$. Pairing the last two retains the strict loss:

$$1 \le d_2 - \frac{A_0 + eA_1}{\kappa A_2} P - \frac{edA_2}{\kappa} r_2 < d_2.$$

The coefficients satisfy the same comparisons used in case B,

$$f_i g_{i+1} \le 1, \qquad f_1 f_2 \le g_1, \qquad g_0 g_2 \le f_0.$$

For the first pair, use the identity

$$(1 - f_0 g_1)(a+c)(1+ae)\kappa\Lambda = (cA_0 - aeM)^2.$$

For the others, (13) gives $C_0 \le \kappa$ and $C_1 + A_0 A_2 d/\kappa \le \Lambda$, proving the two remaining pair bounds. The path bounds follow from $C_0 \le A_0 + s\Lambda$ and

$$C_1 + A_0 A_2 d/\kappa \le tA_0 + M \le t\kappa + M.$$

Use the positive weights v_i of (12). As in case B, $f_i v_i \le v_{i+1}$ and $g_i v_i \le v_{i+2}$, so $v_i \le v_i d_i \le T$ for every i. Now $d_2 > 1$ gives $v_2 < T$, again impossible for the positive weighted average $T = \sum_i r_i v_i$.

Case G. The three cyclic mixed caps give positive slopes α_i and nonnegative slack functions throughout K,

$$s_i(x) = x_{i+1} - \alpha_i x_i - b_i, \qquad b_i = L_{i+1} - \alpha_i U_i.$$

Their halfplanes enclose a triangle. To see this directly, set

$$c_i = a_{i+1} + \alpha_{i+1} a_{i+2} + \alpha_{i+1}\alpha_{i+2} a_i > 0.$$

On the positive plane, expansion gives the constant identity

$$\sum_i c_i s_i(x) = (1 - \alpha_0\alpha_1\alpha_2)D - \sum_i c_i b_i =: C.$$

The slack map on the plane is injective: equal slacks at x, y give $x_{i+1} - y_{i+1} = \alpha_i(x_i - y_i)$, so the coordinate differences have the same sign; their positive weighted sum is zero, so all vanish. Since K contains distinct points and its slacks are nonnegative, $C > 0$. The map $x \mapsto (c_i s_i(x)/C)_i$ is therefore an affine isomorphism onto the plane $\sum_i \lambda_i = 1$. The region T where all slacks are nonnegative is an actual triangle containing K.

The same three mixed caps are empty on T. Any nonconstant cyclic sequence of outside signs has a $+$ followed by a $-$ and is excluded by one of them. The two constant signs are excluded by the plane bounds of Lemma A.7. Thus the original windows cover T. Each coordinate range D_i on T is at least one. The covering bound Theorem 2.1 gives

$$1 \le \sum_i \frac{r_i}{D_i} \le \sum_i r_i = 1.$$

Since every $r_i > 0$, all $D_i = 1$, and the attained ranges are exactly $[0, 1]$. Each strictly interior window contains at most one vertex of T, so covering assigns the three windows bijectively to its vertices. For any one coordinate, the other two vertices are its unique minimum and maximum. Consequently each vertex is a unique extremum of some coordinate. Because K attains all these extrema, it contains all vertices of T. Convexity now gives $K = T$.

Case A. We use equality in the numerical support criterion. Rescale to $y_i = a_i x_i$, so that $\sum_i y_i = D$ and the ranges are $[0, D_i]$ with $D_i = a_i > 0$. Write the rescaled windows as $[\ell_i, h_i]$ and their widths as $\delta_i > 0$. Then $\sum_i \delta_i/D_i = 1$. Fix one minimum and one maximum contact for each coordinate, writing v_i for the chosen $y_i = D_i$ contact. For any $z \in K$, let F consist of these six contacts and z, and put $C = \operatorname{conv} F$.

The six same-sign caps are empty on C. Choose the positive support slopes nearest to one as in Lemma A.4. They satisfy (11). Write the upper and lower slopes as α_i, β_i, and put, with cyclic indices,

$$A_i = 1 + \beta_i\alpha_{i+1}\beta_{i+2}, \qquad B_i = 1 + \alpha_i\beta_{i+1}\alpha_{i+2}, \qquad E_i = A_i B_i,$$
$$S_i = E_i\delta_i + \beta_i B_i(\delta_{i+1} + \alpha_{i+1}\delta_{i+2}) + \alpha_i A_i(\delta_{i+1} + \beta_{i+1}\delta_{i+2}).$$

The numerical support calculation in the proof of Lemma 4.4gives $S_i \ge E_i D_i > 0$ and

$$0 \le \frac{\sum_i E_i\delta_i S_{i+1}S_{i+2} - \prod_i S_i}{\prod_i S_i} = \sum_i \frac{E_i\delta_i}{S_i} - 1 \le \sum_i \frac{\delta_i}{D_i} - 1 = 0.$$

The first inequality is the nonnegative cubic identity in that calculation. Since every $\delta_i > 0$, equality forces $S_i = E_i D_i$ for all i. The two contact bounds used there,

$$A_i\ell_i \le \beta_i(\delta_{i+1} + \alpha_{i+1}\delta_{i+2}), \qquad B_i(D_i - h_i) \le \alpha_i(\delta_{i+1} + \beta_{i+1}\delta_{i+2}),$$

are therefore both equalities.

At v_i, the upper contact bound was obtained by adding the nonnegative slacks of upper row i, lower row $i+1$ and upper row $i+2$, with positive coefficients $1, \alpha_i, \alpha_i\beta_{i+1}$. Its equality forces all three slacks to vanish. In particular, v_i lies on both adjacent upper support lines.

The points v_0, v_1, v_2 are affinely independent. Indeed, for $j \neq i$ the empty upper pair cap gives $y_i(v_j) \leq h_i < D_i = y_i(v_i)$. Thus each is a strict coordinate maximum among the three; none can lie between the other two. Each upper support line contains its two adjacent vertices and is nonconstant on the plane, since its missing third coordinate has a positive plane coefficient. These three lines are exactly the side lines of $\operatorname{conv}\{v_0, v_1, v_2\}$. Their halfplanes containing C consequently enclose that triangle. Hence z lies in it. The maximum contacts were fixed independently of z, so every point of K lies in the same triangle. The reverse inclusion follows from convexity, proving $K = \operatorname{conv}\{v_0, v_1, v_2\}$.

This completes the proof of Lemma A.8. The following calculation gives the exact parameters on that triangle.

Lemma A.9 (The parameters on a triangle)**.** *Every tight cover of a triangle by three strictly interior positive windows has the parameters of family IV.*

Proof. An affine coordinate attains its minimum and maximum at vertices, so a strictly interior window can contain at most one of the triangle's three vertices. Relabeling assigns window i to vertex v_i. Reflect each coordinate to arrange

$$x_i(v_i) = d_i \in (0,1), \qquad x_i(v_{i+1}) = 0, \qquad x_i(v_{i+2}) = 1.$$

Put $G_i = 1 - U_i$ and $r_i = 1 - L_i - G_i$. Then $L_i, G_i, r_i > 0$, $L_i \leq d_i \leq 1 - G_i$, and $\sum_i r_i = 1$. In barycentric coordinates, $x_i = d_i\lambda_i + \lambda_{i+2}$. On the side $\lambda_i = 0$, with $t = \lambda_{i+2}$, the three covered intervals are

$$[L_i, 1 - G_i], \quad [0, 1 - L_{i+1}/d_{i+1}], \quad [G_{i+2}/(1 - d_{i+2}), 1].$$

Consequently, either the latter two intervals meet, or the first bridges their gap. Denote these alternatives by

$$\begin{aligned} M_i: &\quad \frac{L_{i+1}}{d_{i+1}} + \frac{G_{i+2}}{1 - d_{i+2}} \leq 1, \\ B_i: &\quad L_{i+1} + d_{i+1}L_i \leq d_{i+1}, \qquad G_{i+2} + (1 - d_{i+2})G_i \leq 1 - d_{i+2}. \end{aligned}$$

Two useful strict consequences are

$$B_i, M_{i+2} \implies L_i < r_{i+1}, \qquad B_i, M_{i+1} \implies G_i < r_{i+2}. \tag{14}$$

For the first, M_{i+2} and $L_i/d_i > L_i$ give $G_{i+1} < (1 - d_{i+1})(1 - L_i)$; add the first inequality of B_i. The second follows in the same way from $G_i/(1 - d_i) > G_i$ and the second inequality of B_i.

The side choices cannot mix. One bridge and two meetings contradict $L_i + G_i = 1 - r_i$ by (14). With B_i, B_{i+1}, M_{i+2}, those same consequences give $G_i > r_{i+2}$ and $L_{i+1} > r_{i+2}$. But the two bridge inequalities give

$$d_{i+2}L_{i+1} + (1 - d_{i+2})G_i \leq r_{i+2},$$

a contradiction. Hence all sides meet or all sides bridge.

All meetings leave an uncovered point. Indeed, put $u_i = 1 - L_i/d_i$ and $v_i = 1 - G_i/(1 - d_i)$, so $r_i = d_iu_i + (1 - d_i)v_i$ and $u_{i+1} + v_{i+2} \geq 1$. If $u_i \leq v_i$ and $v_{i+1} \leq u_{i+1}$, then $u_i + v_{i+1} \leq r_i + r_{i+1} < 1$, a contradiction. Going around the cycle therefore gives either $u_i < v_i$ for all i, or $v_i < u_i$ for all i. In the first case $u_i < r_i$ and the barycentric point $\lambda_i = r_{i+2}$ satisfies

$$L_i - x_i = (1 - d_i)(r_{i+2} + v_i - 1) > 0$$

for every i. In the second case $v_i < r_i$, and $\lambda_i = r_{i+1}$ satisfies $x_i - U_i = d_i(u_i + r_{i+1} - 1) > 0$ for every i. Both contradict coverage.

Thus all bridges hold. Set

$$P = 1 + \prod_i d_i, \qquad Q = 1 + \prod_i (1 - d_i),$$

and $a_i = 1 - d_{i+1} + d_{i+1}d_{i+2}$, $b_i = d_{i+2} + (1 - d_{i+2})(1 - d_{i+1})$, all positive. Expansion gives the identity

$$\begin{aligned} PQ\left(\sum_i r_i - 1\right) = (P - Q)^2 &+ Q\sum_i a_i(d_i - L_i - d_i L_{i+2}) \\ &+ P\sum_i b_i(1 - d_i - G_i - (1 - d_i)G_{i+1}). \end{aligned}$$

Every term on the right is nonnegative. Tightness forces $P = Q$ and

$$L_i + d_i L_{i+2} = d_i, \qquad G_i + (1 - d_i)G_{i+1} = 1 - d_i.$$

Eliminating around each three-cycle now gives

$$P(1 - L_i) = 1 - d_i + d_i d_{i+2} = Q(1 - G_{i+2}).$$

Hence $L_i = G_{i+2}$. Since $\sum_i r_i = 1$, we have $\sum_i L_i = 1$; thus $L_i = r_{i+1}$ and $G_i = r_{i+2}$. Finally $d_i(1 - L_{i+2}) = L_i$ yields $d_i = r_{i+1}/(1 - r_i)$, precisely (8). □

Proof of Theorem A.2. Delete any zero-width member by closedness and apply Lemma A.1. This gives family I while retaining the deleted member's arbitrary position. Suppose henceforth that all widths are positive.

No width is spent outside the body's range. Indeed, clipping preserves the covering and can only decrease each width; any decrease would contradict Theorem 2.1.

There are three direction cases. If all directions are parallel, independently orient them to the same normalized coordinate. Three intervals cover $[0, 1]$ and have total length one, so their interiors are disjoint and they partition $[0, 1]$ consecutively. This is family II.

If exactly two directions agree after orientation, Lemma A.5 gives family V.

Otherwise the directions are pairwise independent. Their unique linear dependence has no zero coefficient. Reflecting coordinates makes its three coefficients positive. All positive endpoints are now within their actual ranges. If an endpoint is a support level, apply Lemma A.6; otherwise apply Lemmas A.8 and A.9. This yields families II–IV and exhausts necessity.

For sufficiency, family I follows from Lemma A.1, and family V follows by subdividing one of its windows. For family II, a point missed by the first window has $x_0 > r$, hence $x_1 > r$; if the second also misses it, then $x_1 > s$ and $x_2 > s$, so the third contains it. For family III, every body point with $x_0 > r$ belongs to $rT + (1 - r)K$, and the pulled-forward two-window form covers this smaller triangle.

For family IV write a point as $\sum_i \lambda_i v_i$, where $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$, and put $z_i = r_i\lambda_i$. The coordinate formula in (8) shows that membership in window i is equivalent to

$$-r_i r_{i+1} \leq z_{i+2} - z_{i+1} \leq r_i r_{i+2}.$$

Suppose every window misses the point, and order the three values as $z_a \geq z_b \geq z_c$. The missed windows corresponding to the pairs (a, b) and (b, c) imply, respectively,

$$z_a - z_b > r_c r_a, \qquad z_b - z_c > r_a r_b.$$

It follows that

$$1 = \sum_i \frac{z_i}{r_i} > \frac{z_c + r_a r_b + r_a r_c}{r_a} + \frac{z_c + r_a r_b}{r_b} + \frac{z_c}{r_c} = 1 + z_c \sum_i \frac{1}{r_i} \geq 1,$$

a contradiction. Ties among the z_i already put the point in the window for that pair. All displayed families have total relative width one, as their endpoint formulas show. This completes sufficiency. □

B Finite-body reductions and the planar vertex bound

B.1 Normalized finite hulls in every dimension

Lemma B.1. *Every convex body in $\mathbb{R}^d$ has an affine image K' satisfying*

$$\Delta_d = \operatorname{conv}(0, e_1, \ldots, e_d) \subset K' \subset [-1, 1]^d.$$

The simultaneous affine image of its planks has the same relative widths and the same uncovered-volume fraction.

Proof. Fix an interior point A and maximize $|\det(v_1 - A, \ldots, v_d - A)|$ over $v_i \in K$. Compactness supplies a maximum and an interior ball makes it positive. Send A to 0 and v_i to e_i. Replacing the ith vector by any body point multiplies the determinant by that point's ith coordinate; maximality bounds its absolute value by one. Convexity gives the inner simplex. Linear changes multiply both volumes by the same Jacobian, and the numerator and denominator of each relative width transform together. □

For a normalized body, write $s_i = \min_K u_i \cdot x$ and $D_i = \max_K u_i \cdot x - s_i$, where $u_i \in S^{d-1}$. Describe its planks by

$$s_i + \ell_i D_i \leq u_i \cdot x \leq s_i + h_i D_i, \qquad 0 \leq \ell_i \leq h_i \leq 1, \quad \sum_i (h_i - \ell_i) \leq 1.$$

Clamping physical endpoints to the support interval changes coverage only on supporting hyperplanes and cannot increase relative width. Thus it cannot increase μ_d.

For each integer $N \geq 0$, consider

$$K_N = \operatorname{conv}(0, e_1, \ldots, e_d, X_1, \ldots, X_N), \qquad X_j \in [-1, 1]^d. \tag{15}$$

Let Ω_N consist of these point tuples, unit directions and legal normalized windows, with the relative topology of their product space. All hulls contain Δ_d, so every directional width is positive.

Proposition B.2. *Fix $d, m \geq 1$. If a strict counterexample exists, then for some finite N the strict-failure set $E_N \subset \Omega_N$ is nonempty and relatively open. Suppose a probability law ν_N gives positive measure to every nonempty relatively open subset of Ω_N, for every N, and choose weights $p_N > 0$ with $\sum_{N \geq 0} p_N = 1$. Then*

$$\sum_{N \geq 0} p_N \nu_N(E_N) > 0.$$

Proof. Normalize the counterexample by Lemma B.1 and clamp its windows. Its defect remains negative. A finite ε-net $S \subset K$ gives

$$\operatorname{conv}(\Delta_d \cup S) \subset K, \qquad d_H\big(\operatorname{conv}(\Delta_d \cup S), K\big) \leq \varepsilon.$$

On the family $\Delta_d \subset K \subset [-1,1]^d$, support values and widths vary continuously, and widths have a uniform positive lower bound. Volume and uncovered volume are also jointly continuous in the body, directions and endpoints. Indeed, off the boundary of the limiting body and its finitely many plank hyperplanes, the corresponding indicators converge pointwise. A fixed interior ball of Δ_d gives stability of interior membership under Hausdorff convergence, and all indicators are bounded by that of the cube. Dominated convergence applies, since these boundaries have volume zero. The volume denominator is uniformly positive.

Consequently a sufficiently fine net retains the strict failure. The map from a point tuple to its hull is 1-Lipschitz for the maximum point distance and the Hausdorff metric: use the same convex coefficients for corresponding points. Continuity of μ_d therefore makes E_N relatively open. Full support gives $\nu_N(E_N) > 0$; the corresponding positive weighted term proves the assertion. □

The proposition supplies no fixed point-count bound or useful lower bound on the failure probability. Full support is a hypothesis about the *joint* law of points, directions and windows at every finite count; positivity of every count weight is a separate hypothesis. A fixed point budget, finite deficit grid, positive width floor or finite-precision implementation has no such completeness guarantee.

In the plane a stronger finite reduction is available. For the rest of this appendix put $\mu = \mu_2$, write

$$P_i = \{x : L_i \le u_i \cdot x \le H_i\}, \qquad a_i = \min_K u_i \cdot x, \quad b_i = \max_K u_i \cdot x, \quad D_i = b_i - a_i,$$

and set $U = \bigcup_i P_i$, $V = |K|$ and $G = |K \setminus U|$. Put $r_i = (H_i - L_i)/D_i$. Thus $q = G/V$, $R = \sum_i r_i$ and $\mu = q - (1-R)_+^2$.

We prove Theorem 10.3, stated in Section 10.

The replacement may change the planks. The theorem bounds the number of polygon vertices, whereas (15) counts input points in addition to a fixed simplex. Neither reduction gives a finite list of numerical tests.

B.2 A polygonal global minimum

Lemma B.3. *If an m-plank counterexample exists, the defect has a negative global minimum, over all planar convex bodies and m-plank configurations with $R \le 1$. A minimum is attained by a polygon.*

Proof. Clamp both endpoints of every plank to its support interval $[a_i, b_i]$. The clamp is monotone and 1-Lipschitz, so R cannot increase. Coverage inside K changes only on support lines, which have area zero. In particular, a disjoint window becomes a zero-width support plank: we retain exactly m planks. This operation cannot increase μ.

By Lemma B.1 we may work with all compact convex K satisfying $\Delta_2 \subset K \subset [-1,1]^2$. This is a compact family in the Hausdorff metric: it is a closed subspace of the compact space of nonempty compact subsets of the square. No bound on the number of vertices is needed. Add unit normals $u_i \in S^1$ and normalized windows

$$0 \le \ell_i \le h_i \le 1, \qquad \sum_i (h_i - \ell_i) \le 1,$$

with physical levels $a_i + \ell_i D_i$ and $a_i + h_i D_i$. Support values are continuous and the widths are uniformly positive, because every body contains Δ_2.

Area and uncovered area vary continuously: the indicators converge off the boundary of the limiting convex body and its finitely many plank lines, and all are bounded by the indicator of the square. To see stability of interior membership, translate a fixed interior ball of Δ_2 to the origin. If $\overline{B}(0,r) \subset L$ and $d_H(K,L) < r(1-t)$, then $tK \subset L$, for $0 < t < 1$: choose $y \in L$

close to $x \in K$ and write $tx = ty + (1-t)t(x-y)/(1-t)$. The last point lies in the inner ball. Dominated convergence applies. The area denominator is at least $1/2$. Thus the defect attains a negative minimum on this compact family.

It is global. Every negative competitor admits a representative in the compact family with no larger defect, by clamping and affine normalization. A nonnegative competitor cannot improve the minimum. Therefore subsequent body variations need not preserve the normalization.

Finally apply the width-preserving polygon replacement of Section B.3 to this minimum. It cannot increase the defect, so the resulting polygon is another global minimum. This step uses no preliminary bound on the number of sides and allows parallel directions. □

B.3 Polygonal replacement

This construction turns the global minimum into a polygon. It requires only that the defect be negative. Choose at most $2m$ support witnesses S and trim to $C = \operatorname{conv}(S \cup (K \cap U))$. The widths stay fixed and only uncovered area is removed. If $D = |K| - |C|$, the unnormalized defect satisfies

$$|C \setminus U| - \delta^2|C| = G - \delta^2 V - (1-\delta^2)D < 0.$$

Hence $|C| > 0$, and $(G-D)/(V-D) \le G/V$. Every uncovered extreme point of C belongs to S.

For each choice of a lower or upper outside halfplane for each plank, let Q be their closed intersection. Ignore lower-dimensional Q. An extreme point of $C \cap Q$ in $\operatorname{int} Q$ is an extreme point of C, hence a member of S. Any other extreme point is an endpoint of $(C \cap Q) \cap L$ for one of the at most m wall lines L. Thus each positive-area $C \cap Q$ is a polygon.

Retain the supporting halfplanes of its edges not on ∂Q. They contain all of C: a short segment from an edge-interior point in $\operatorname{int} Q$ to a point of C on the wrong side would contradict the support property in $C \cap Q$. Along with Q these halfplanes define exactly $C \cap Q$. If that intersection has zero area or is empty, separate $\operatorname{int} C$ from Q and retain one halfplane containing C whose intersection with Q has zero area.

Intersect all retained halfplanes with the $2m$ original support halfplanes and with a closed square containing C. The result O is bounded. It contains C and its witnesses, preserves all widths, and has exactly the same uncovered area as C, by the sign-cell decomposition up to null wall lines. Therefore $q(O) \le q(C) \le q(K)$.

There are finitely many cells and finitely many retained halfplanes. That finiteness is all the global-minimum construction needs; the final $4m$ estimate comes from the geometric variation and section-endpoint count.

B.4 Uncovered vertices

Fix a polygonal global negative minimum. Since $R < 1$, $0 \le q < \delta^2 \le 1$. There are $2m$ indexed support faces

$$F_i^- = K \cap \{u_i \cdot x = a_i\}, \qquad F_i^+ = K \cap \{u_i \cdot x = b_i\}.$$

A singleton support face determines one vertex. Repeated faces only overcount and cause no difficulty.

Lemma B.4. *Every uncovered vertex is a singleton support face for at least one indexed direction. In particular, there are at most $2m$ uncovered vertices.*

Proof. Suppose an uncovered vertex v is not any singleton support face. Choose one point in every support face, each different from v, and call their set S. The compact convex set

$$C = \operatorname{conv}(S \cup (K \cap U))$$

preserves every support width and removes only uncovered area. It excludes v: an extreme point in a convex hull must belong to its generating set. It has positive area, since it contains the covered part of K, whose area is $V - G > 0$.

Nested positive-area compact convex sets with the same area coincide. Thus $D = V - |C| > 0$, and

$$\frac{|C \setminus U|}{|C|} = \frac{G - D}{V - D} < \frac{G}{V},$$

where strictness follows from $G < V$. Widths, hence δ, stay fixed, contradicting minimality. Each indexed singleton face supplies at most one vertex, proving the count. □

B.5 No whole edge is covered

First move every zero-width plank outside K. Its contribution to R is zero, and changing a finite union of lines does not change area. Thus the same body and modified configuration remain a global minimum.

Lemma B.5. *Every positive-width window at a negative global minimum satisfies $a_i < L_i < H_i < b_i$.*

Proof. Suppose $L_i \le a_i$. Set $\theta = 1 - r_i$; then $0 < \theta < 1$, since $0 < r_i \le R < 1$. Choose a point $p \in K$ with $u_i \cdot p = b_i$, and contract K about p by θ. The resulting body C lies in K and satisfies

$$\min_C u_i \cdot x = a_i + r_i D_i \ge H_i.$$

Collapse the ith window to its upper endpoint. Every other window stays fixed. All directional widths scale by θ, so the new budget is $R' = (R - r_i)/\theta$ and its deficit is $1 - R' = (1 - R)/\theta$. The new uncovered set is contained in the old one, while $|C| = \theta^2 V$. Consequently its defect is at most

$$\frac{G}{\theta^2 V} - \frac{(1 - R)^2}{\theta^2} = \frac{\mu}{\theta^2} < \mu,$$

contradicting global minimality. Reverse the direction to exclude $H_i \ge b_i$. □

Lemma B.6. *If $q > 0$, no polygon edge is fully covered.*

Proof. Suppose an edge e is covered. Take orthonormal coordinates with its supporting line $y = 0$ and $K \subset \{y \le 0\}$. Remove this edge's inequality from an irredundant halfplane description, and intersect the remaining halfplanes with a large closed disk containing K in its interior. The resulting compact convex set L satisfies

$$K = L \cap \{y \le 0\}, \qquad K_\varepsilon = L \cap \{y \le \varepsilon\}.$$

An interior point of e satisfies every other row strictly, so L extends across $y = 0$. Its sections near zero are intervals with endpoints converging to those of e. Their total length $\sigma(y)$ tends to $|e|$.

No plank boundary coincides with this supporting line: positive windows are strictly interior by Lemma B.5, and null windows were moved outside. Thus only finitely many points of e lie on plank boundaries. Every other point is strictly inside a covering plank. Dominated convergence on a fixed bounded interval shows that the uncovered section length $\rho(y)$ tends to zero. Fubini gives

$$\Delta V = |e|\varepsilon + o(\varepsilon), \qquad \Delta G = o(\varepsilon).$$

Enlargement increases directional widths and therefore cannot increase R. Global minimality then gives $(G + \Delta G)/(V + \Delta V) \ge G/V$, hence $\Delta G \ge q \Delta V$. Division by ε and passage to the limit contradict $q|e| > 0$. □

B.6 Count section endpoints

Lemma B.7. *If a strip contains three vertices of a convex polygon, it contains an entire edge of the polygon.*

Proof. Choose coordinates making the strip vertical. Every extreme point is an upper or lower endpoint of its vertical section: otherwise it lies strictly between two points of the polygon. It suffices to show that two upper extreme points u, v force an entire edge between their horizontal coordinates. Distinct upper points have distinct horizontal coordinates; write $u_x < v_x$.

Choose the upper section endpoint p above $(u_x+v_x)/2$. In an irredundant halfplane description, some row active at p has positive vertical coefficient. Otherwise a sufficiently small upward displacement would satisfy every row, contradicting maximality of p. Let F be the edge defined by this row.

Suppose $z \in F$ lies to the left of u. The segment $[z, p]$ crosses the vertical line through u at an interior point p'. The row's positive vertical coefficient and the polygon's halfplane inequality imply $u_y \le p'_y$. Since u is the upper endpoint, $p'_y \le u_y$ as well. Thus $u = p'$, contradicting extremality. Similarly no point of F lies to the right of v. Hence F lies inside the strip. Reflection gives the same result for two lower endpoints; among three vertices two have the same type. □

Proof of Theorem 10.3. Choose the polygonal global negative minimum. If $q > 0$, move the null windows outside as above. By Lemmas B.6 and B.7, each plank contains at most two vertices. There are therefore at most $2m$ covered vertices. Lemma B.4 bounds the uncovered vertices by $2m$, giving $4m$ in total.

If $q = 0$, closedness of U implies $K \subset U$. Otherwise an omitted point has a neighborhood outside U meeting the positive-area convex body in positive area. Choose one witness from each of the $2m$ support faces and replace K by their convex hull C. This preserves every width and full coverage, so the defect is unchanged.

The hull cannot be a segment $[A, B]$. On that segment each plank form has span $D_i > 0$, so its covering interval has relative length at most $(H_i - L_i)/D_i$. Such intervals cannot cover $[A, B]$ when their lengths sum to $R < 1$. Nor can a point preserve positive widths. Thus C has positive area and at most $2m$ vertices. In both cases its defect is no greater than the original counterexample's defect. □

References

[1] A. Akopyan and A. Polyanskii, *A short proof of Ball's plank theorem*, preprint, 2026, arXiv:2609.36163.

[2] G. Ambrus, *Appendix: Plank problems*, manuscript, 2010, `https://users.renyi.hu/~ambrus/appendix.pdf`.

[3] G. Ambrus, M. Corbo, D. N. Goto, D. Leitz and K. Mawhinney, *On plank coverings of cube annuli*, preprint, 2026, arXiv:2610.18770.

[4] E. Bakaev and A. Polyanskii, *Optimal partial plank coverings*, preprint, 2026, arXiv:2607.27483.

[5] E. Bakaev and A. Yehudayoff, *A note on the affine plank conjecture*, preprint, 2026, arXiv:2602.20290.

[6] E. Bakaev and A. Yehudayoff, *Volume stability of plank covers*, preprint, 2026, arXiv:2609.30409v2.

[7] K. Ball, *The plank problem for symmetric bodies*, Invent. Math. **104** (1991), 535–543.

[8] T. Bang, *A solution of the "plank problem"*, Proc. Amer. Math. Soc. **2** (1951), no. 6, 990–993.

[9] T. Bang, *Some remarks on the union of convex bodies*, Tolfte Skandinaviska Matematikerkongressen, Lund 1953, Lunds Universitets Matematiska Institution, Lund, 1954, 5–11.

[10] K. Bezdek, *Tarski's plank problem revisited*, in *Geometry—Intuitive, Discrete, and Convex*, Bolyai Soc. Math. Stud. **24**, Springer, 2013, 45–64; arXiv:0903.4637, 2009.

[11] R. J. Gardner, *Relative width measures and the plank problem*, Pacific J. Math. **135** (1988), no. 2, 299–312.

[12] R. J. Gardner, *The Brunn–Minkowski inequality*, Bull. Amer. Math. Soc. **39** (2002), 355–405.

[13] H. F. Hunter, *Some special cases of Bang's inequality*, Proc. Amer. Math. Soc. **117** (1993), no. 3, 819–821.

[14] J. Matoušek, *Using the Borsuk–Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry*, Universitext, Springer, Berlin, 2003.

[15] W. Verreault, *Plank theorems and their applications: a survey*, Bull. Lond. Math. Soc. **58** (2026), no. 1, e70230, doi:10.1112/blms.70230; arXiv:2203.05540v2.

[16] Y. Zhang and R. Ding, *A note about Bezdek's conjecture on covering an annulus by strips*, Electron. J. Combin. **15** (2008), no. 1, N19, doi:10.37236/894.